Quotients of $\mathbb{G}_a$ Actions

David R. Finston
Brooklyn College CUNY

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Affine Varieties:

Work over $\mathbb{C}$ although most everything in this section can be formulated for an arbitrary algebraically closed field.

An algebraic set V in $\mathbb{C}^n$ is the collection of common zeros of a subset $S \subset \mathbb{C}[x_1, \ldots, x_n]$ write $V = \mathcal{V}(S))$

$V = \mathcal{V}(I))$ where I is the ideal generated by S.

Hilbert Basis Theorem: S can be taken to be finite.

If I is prime then V is called an affine variety.

Given an algebraic set V, the set

$$\mathcal{I}(V) = \{f \in \mathbb{C}[x_1, \ldots, x_n] : f(v) = 0 for all v \in V\}$$

is an ideal.

Hilbert Nullstellensatz If $V = \mathcal{V}(I)$, then

$$\mathcal{I}(\mathcal{V}) = \cup_{j \geq 0}\{f : f^j \in I\} := \sqrt{I}.$$

If I is prime, then $\mathcal{I}(\mathcal{V}(I)) = I$.

The algebraic notion of primeness:

For ideals I_1, I_2 the product $I_1 I_2 \subset I$ implies $I_1 \subset I$ or $I_2 \subset I$

translates to V **is irreducible**:

For algebraic sets V_1, V_2 if $V = V_1 \cup V_2$ then $V = V_1$ or $V = V_2$.

The algebraic subsets of $\mathbb{C}^n$ satisfy the axioms for the closed sets of a topology, called the **Zariski topology** on $\mathbb{C}^n$.

V is a variety iff V is an irreducible closed set for the Zariski topology.

Let V be a variety in $\mathbb{C}^n$ with ideal I. The well defined algebraic functions on V are represented by restrictions to V of polynomials functions on $\mathbb{C}^n$.

$f|_V = g|_V$ iff $(f-g)|_V \equiv 0$ iff $f - g \in I$ (Hilbert Nullstellensatz).

The ring $\mathcal{O}(V)$ of functions well defined everywhere on V is isomorphic to the **affine (integral) domain** $\mathbb{C}[x_1, \ldots, x_n]/I$.

Its quotient field $\mathbb{C}(V)$ consists of functions defined on some (Zariski) open subset of V.

Equivalence of categories:

Finitely generated $\mathbb{C}-$algebra domains $\Longleftrightarrow$ Complex affine varieties

Algebra homomorphisms$\Longleftrightarrow$ Mappings represented coordinate-wise by polynomial functions.

$$A \cong \mathbb{C}[x_1, \ldots, x_n]/I \cong O(\mathcal{V}(I)) \leftrightarrow \textbf{Spec } A := \mathcal{V}(I) \subset \mathbb{C}^n$$

$$V \cong W \text{ iff } \mathcal{O}(V) \cong \mathcal{O}(W)$$

Here $V \cong W$ means the existence of a mapping with two sided inverse.

Cancellation

Example: The cylinder over a variety $V = \mathcal{V}(I) \subset \mathbb{C}^n$ is the variety

$$\begin{aligned} V \times \mathbb{C} &= \mathcal{V}(I\mathbb{C}[x_1, \ldots, x_n, x_{n+1}]) \subset \mathbb{C}^{n+1}. \\ \mathcal{O}(V \times \mathbb{C}) &\cong \mathcal{O}(V)[x_{n+1}] \end{aligned}$$

The general question: If $V \times \mathbb{C} \cong W \times \mathbb{C}$ is $V \cong W$? (Equivalently, for affine domains A, B and indeterminate t satisfying $A[t] \cong B[t]$ is $A \cong B$?)

has a negative answer. But

Open Problem: For n>2 if $V \times \mathbb{C} \cong \mathbb{C}^{n+1}$ is $V \cong \mathbb{C}^n$?

Failure of cancellation: The affine varieties $D_n : y^2 - 2x^n z - 1 = 0$ satisfy

1. $D_n \cong D_m$ if and only if $n = m$

2. $D_n \times \mathbb{C} \cong D_m \times \mathbb{C}$ for every pair (n, m).

The key to understanding both 1) and 2) is that D_n admits a fixed point free $\mathbb{G}_a$ (=$(\mathbb{C}, +)$) action by automorphisms: $\mathbb{G}_a \times D_n \xrightarrow{\sigma} D_n$:

$$(t, (x, y, z)) \overset{\sigma}{\mapsto} (x, y + tx^n, z + ty + \frac{t^2}{2}x^n)$$

It looks exponential.

On $\mathcal{O}(D_n) \cong \mathbb{C}[x,y,z]/(y^2 - 2x^n z - 1)$ the assignment

$$(t,x) \longmapsto x, (t,y) \longmapsto y + tx^n, (t,z) \longmapsto z + ty + \frac{t^2}{2}x^n$$

extends to an action $\hat{\sigma}$ of $\mathbb{G}_a$ on $\mathcal{O}(D_n)$ by algebra automorphisms.

$\delta(f) := \frac{\hat{\sigma}_t(f)-f}{t}|_{t=0}$ is the **locally nilpotent derivation** (LND) $\delta = x^n \frac{\partial}{\partial y} + y\frac{\partial}{\partial z}$ of $\mathcal{O}(D_n)$ and $\hat{\sigma}_t = \exp(t\delta)$.

Tangent vector to $\mathbb{G}_a$ orbit of (a,b,c) is never 0 (no fixed points):

$$\begin{aligned} &< \delta(x)(a,b,c), \delta(y)(a,b,c), \delta(z)(a,b,c) >=< 0, a^n, b > \\ &\neq 0 \text{ (recall } b^2 - 2a^n c = 1). \end{aligned}$$

For any affine variety V obtain 1-1 correspondence $\hat{\sigma} \to \delta \to \exp(t\delta) = \hat{\sigma}$ between LNDs of $\mathcal{O}(V)$ and $\mathbb{G}_a$ actions on V.

For $n \geq 2$ the $\mathbb{G}_a$ action on D_n is essentially unique. More precisely,

$$\text{ker}(\delta) = \mathbb{C}[x] \text{ for every } \delta \in \text{LND}(\mathcal{O}(D_n)).$$

For $n = 1$ symmetry between x and z shows $\bigcap_{\delta} \text{ker}(\delta) = \mathbb{C}$.

$\cap\{\text{ker}(\delta) : \delta \in \text{LND}(\mathcal{O}(V)\}$ is an isomorphy invariant of affine varieties (characteristic subalgebra): Makar-Limanov invariant.

Thus $D_1 \not\cong D_n$ for $n \geq 2$. The fact that an isomorphism between D_n and D_m would have to carry $\mathbb{C}[x]$ to $\mathbb{C}[x]$ is so restrictive as to exclude any if $m \neq n$.

But why are the cylinders isomorphic?

D_n has a cover by open subsets for the Zariski topology which are themselves $\mathbb{G}_a$ stable affine varieties:

$V_+ := D_n - \mathcal{V}(x, y-1)$, $V_- := D_n - \mathcal{V}(x, y+1)$ cover and their complements are $\mathbb{G}_a$ stable since $\sigma_t(0, \pm 1, z) = (0, \pm 1, z \pm t)$.

$s_+ := \frac{y+1}{x^n} = \frac{2z}{y-1}$ is globally defined on V_+ and $s_- := \frac{y-1}{x^n} = \frac{2z}{y+1}$ is globally defined on V_-. Also y and z can be recovered from x and $s_\pm$

$\mathcal{O}(D_n)[s_+] = \mathbb{C}[x, s_+] = \mathcal{O}(V_+)$, $\mathcal{O}(D_n)[s_-] = \mathbb{C}[x, s_-] = \mathcal{O}(V_-)$

So $V_\pm \cong \mathbb{C}^2$ with coordinates $x, s_\pm$ respectively. Moreover,

$$\hat{\sigma}_t s_\pm = \frac{\hat{\sigma}_t(y \pm 1)}{\hat{\sigma}_t x^n} = \frac{y \pm 1 + t x^n}{x^n} = s_\pm + t$$

$\mathbb{G}_a$ action on V_+ (resp.V_-) is a simple translation: $\sigma_t(x, s_\pm) = (x, s_\pm + t)$.

The morphisms of affine varieties $V_\pm \overset{pr_x}{\to} \mathbb{C}^1$ have fibers parametrized by $s_\pm$

and glue to a morphism $D_n \overset{pr_x}{\to} \mathbb{C}^1$ with a disconnected fiber over the origin:

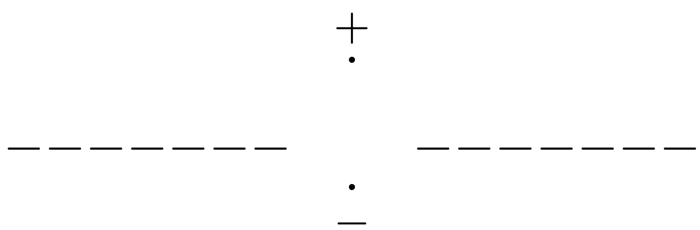
Recall $x^n z = y^2 - 1$ so that $pr_x^{-1}(0) = \{(0, 1, z)\} \cup \{(0, -1, z)\}$.

Action on D_n is **locally trivial for the Zariski topology** (i.e. locally a translation).

What is its quotient (geometric structure on space of orbits)?

On V_+ (resp. V_-) the space of orbits is identified with $\mathbb{C}^1$ with coordinate x.

Replace the origin with 2 points $0^{\pm}$ to obtain the space of orbits as the non-separated scheme



$= \mathbb{C} \amalg \mathbb{C}/\{\alpha \sim \alpha : x \neq 0\} := \tilde{\mathbb{C}} = \mathbb{C}^+ \cup \mathbb{C}^-$.

$D_n \cong$**Spec** $\mathbb{C}[x, s_+]\amalg$ **Spec** $\mathbb{C}[x, s_-]/\{(\alpha, \beta) \sim (\alpha, \beta + \frac{2}{x^n}) : \alpha \neq 0\}$.

$D_n \xrightarrow{\pi} \tilde{\mathbb{C}}$ where $\pi|_{V^\pm} : V^\pm \longrightarrow \mathbb{C}^\pm$ are the two projections.

More precisely, $D_n \xrightarrow{\pi} \widetilde{\mathbb{C}}$ is a (nontrivial) principal $\mathbb{G}_a$ bundle. The gluing data $\frac{2}{x^n} \in \check{H}^1(\widetilde{\mathbb{C}}, \mathcal{O}(\widetilde{\mathbb{C}}))$ a cohomology group that classifies the principal $\mathbb{G}_a$ bundles over $\widetilde{\mathbb{C}}$

$\widetilde{\mathbb{C}}$ is not affine.

Theorem (Serre) If W is an affine variety then every principal $\mathbb{G}_a$ bundle over W is trivial, i.e.

If $V \to W$ is a **principal** $\mathbb{G}_a$ **bundle** then $V \cong W \times \mathbb{G}_a$ with the group acting trivially on W and by translation on itself.

A scheme X is a space covered by affine varieties $X = \cup_{i=1}^{m} X_i$ so that the rings $\mathcal{O}(X_i)$ agree over the intersections of the X_i to get a sheaf of rings.

$\widetilde{\mathbb{C}}$ has the cover by two copies of $\mathbb{C}$ glued over the complement of their origins.

$\check{H}^1(X, \mathcal{O}(X))$ classifies the principal $\mathbb{G}_a$ bundles $Y \xrightarrow{\pi} X$: For a given affine open cover $X = \cup_{i=1}^{m} X_i$, by Serre,

$$\begin{aligned} Y_i \quad &: \quad = \pi^{-1}(X_i) \cong X_i \times \mathbb{G}_a \\ \mathcal{O}(Y_i) \quad &\cong \quad \mathcal{O}(X_i)[s_i] \end{aligned}$$

The $\binom{m}{2}$ tuple $(s_i - s_j)_{i<j} \in Z^1(X.\mathcal{O}(X))$ is a 1-cocycle for Čech cohomology. Coboundaries correspond to the trivial bundle $X \times \mathbb{G}_a$.

Conversely, for an $\begin{pmatrix} m \\ 2 \end{pmatrix}$ tuple with entries in $\mathcal{O}(X_i \cap X_j)$ the coboundary map measures how the s_i coordinates restrict on overlaps of 3 $X_i's$. That we have a cocycle says that they agree in every way they can restrict so that $X_i \times \mathbb{G}_a$ glue to a bundle over X.

Serre's theorem is that if X is affine then every cocycle is a coboundary. By the sheaf property, the s_i glue together to a global coordinate s and trivial bundle $\mathcal{O}(Y) = \mathcal{O}(X)[s]$.

Because of the pole at $x = 0$ in the cocycle $\frac{2}{x^n}$, this is not the case for s_+ and s_- on D_n, hence the **local** in local triviality, i.e. the bundle $D_n \stackrel{\pi_n}{\to} \widetilde{\mathbb{C}}$ is not trivial.

We can use local triviality and Serre's theorem to construct cancellation counterexamples.

Look at the affine variety

$$D_m \times_{\widetilde{\mathbb{C}}} D_n := \{(a,b) \in D_m \times D_n : \pi_m(a) = \pi_n(b)\}.$$

The two projections

$$\begin{array}{ccc} & D_m \times_{\widetilde{\mathbb{C}}} D_n & \\ \swarrow & & \searrow \\ D_m & & D_n \end{array}$$

are $\mathbb{G}_a$ bundles with cocycles $\frac{2}{x^n} \in \check{H}^1(D_m, \mathcal{O}(D_m))$ and $\frac{2}{x^m} \in \check{H}^1(D_n, \mathcal{O}(D_n))$. These are the base extensions corresponding to the cover of $D_m = V_+^m \cup V_-^m$ glued via $\frac{2}{x^n}$ (resp. $D_n = V_+^n \cup V_-^n$ glued via $\frac{2}{x^m}$).

Danielewski fiber product trick: Since D_m and D_n are affine

$$D_m \times \mathbb{C} \cong D_m \times_{\widetilde{\mathbb{C}}} D_n \cong D_n \times \mathbb{C}$$

We are far removed from $\mathbb{C}^n$, the $\mathcal{O}(D_n)$ are not even UFDs. There are threefold UFD counterexamples, but again not $\mathbb{C}^3$ (yet).

$\mathbb{G}_a$ Bundles over a Quasiaffine Quadric Fourfold

Danielewski fiber product trick: If two affine varieties are total spaces of principal $\mathbb{G}_a$ bundles over the same base then they have isomorphic cylinders, so search for locally trivial $\mathbb{G}_a$ actions on a $\mathbb{C}^n$ and on another affine variety X with the same quotient W. Necessarily W will be strictly quasiaffine (i.e. a proper open subset of an affine variety, which has no structure of affine variety itself) otherwise by Serre, $\mathbb{C}^n$ and X are both isomorphic to $W \times \mathbb{C}$.

Take $\delta := x_1 \frac{\partial}{\partial x_2} + y_1 \frac{\partial}{\partial y_2} + (1 + x_1 y_2 - x_2 y_1) \frac{\partial}{\partial z}$ which is a locally nilpotent derivation of $\mathbb{C}[x_1, y_1, x_2, y_2, z]$.

The associated action σ is

$$\sigma_t(a_1, b_1, a_2, b_2, c) = (a_1, b_1, a_2 + ta_1, b_2 + tb_2, c + t(1 + a_1b_2 - a_2b_1))$$

$$\hat{\sigma}_t \frac{x_2}{x_1} = \frac{x_2}{x_1} + t,\ \ \hat{\sigma}_t \frac{y_2}{y_1} = \frac{y_2}{y_1} + t,\ \ \hat{\sigma}_t \frac{z}{(1 + x_1y_2 - x_2y_1)} = \frac{z}{(1 + x_1y_2 - x_2y_1)} + t.$$

Obtain local coordinates translated by the action:

$$s_1 := \frac{x_2}{x_1},\ \ s_2 :== \frac{y_2}{y_1},\ \ s_3 := \frac{z}{(1 + x_1y_2 - x_2y_1)}.$$

For each point x of $\mathbb{C}^5$ at least one $s_i(x)$ is defined. The action is locally trivial.

The orbit space (geometric quotient) W is the complement of a subvariety of **codimension 2 in a smooth affine quadric**

$$\begin{aligned} Y &:= \mathcal{V}(c_1 c_5 - c_2 c_4 + c_3(1 + c_3)) \subset \mathbb{C}^5 \\ W &:= Y - \mathcal{V}(c_1, c_2, (1 + c_3)). \end{aligned}$$

Here the c_i generate the ring of $\mathbb{G}_a$ invariant functions with

$$\begin{aligned} c_1 &:= x_1,\ c_2 := y_1,\ c_3 := x_1 y_2 - x_2 y_1, \\ c_4 &:= x_1 z - x_2(1 + x_1 y_2 - x_2 y_1), \\ c_5 &:= y_1 z - y_2(1 + x_1 y_2 - x_2 y_1) \end{aligned}$$

By Hartog W is not affine, $\mathcal{O}(W) = \mathcal{O}(Y)$ but $W \subsetneq Y$.

W has the cover by affine open subsets

$$
\begin{aligned}
W_1 \quad &: \quad = W - \mathcal{V}(c_1) \cong \mathbf{Spec}\ \mathcal{O}(W)[\frac{1}{c_1}], \\
W_2 \quad &: \quad = W - \mathcal{V}(c_2) \cong \mathbf{Spec}\ \mathcal{O}(W)[\frac{1}{c_2}] \\
W_3 \quad &: \quad = W - \mathcal{V}(1 + c_3) \cong \mathbf{Spec}\ \mathcal{O}(W)[\frac{1}{1 + c_3}]
\end{aligned}
$$

The Čech cocycle for the bundle is

$$
(s_3 - s_2, s_3 - s_1, s_2 - s_1) = (\frac{c_5}{c_2(\ 1 + c_3)}, \frac{c_4}{c_1(\ 1 + c_3)}, \frac{c_3}{c_1 c_2}).
$$

Now we need to find other affine varieties admitting a locally trivial $\mathbb{G}_a$ action with quotient W (affine total spaces of $\mathbb{G}_a$ bundles over W).

Work backwards. We want a cancellation context, i.e. a variety X so that $X \times \mathbb{C}^5 \cong X \times_W \mathbb{C}^5 \cong \mathbb{C}^6$.

Suppose we have one. Independent of the affineness of X, because of the affineness of $\mathbb{C}^5$ the base extension

$$\mathbb{C}^5 \times_W X \cong \mathbb{C}^5 \times \mathbb{C}^1 \cong \mathbf{Spec}\ \mathbb{C}[x_1, y_1, x_2, y_2, z, u]$$

and it inherits the $\mathbb{G}_a$ action $t \cdot (a, \lambda) = (\sigma_t(a), \lambda)$ where $a \in \mathbb{C}^5$ and $\lambda \in \mathbb{C}$. This action is generated by an extension of δ to another locally nilpotent derivation $\hat{\delta}$ of $\mathbb{C}[x_1, y_1, x_2, y_2, z, u]$. A straightforward calculation shows that local nilpotency forces $\hat{\delta}(u) \in \mathbb{C}[x_1, y_1, x_2, y_2, z]$.

If X is affine then $\mathbb{C}^5 \times_W X \cong X \times \mathbb{C}^1$ as a bundle over X. This means that $\mathcal{O}(X \times \mathbb{C}^1) = \mathcal{O}(X)[t]$, $\hat{\delta}(t) \in \mathbb{C}^*$, and $\mathcal{O}(X) = \ker \hat{\delta}$. So $X = \mathbf{Spec} \ker \hat{\delta}$.

Conversely, each extension $\hat{\delta}$ as above gives rise to a $\mathbb{G}_a$ bundle over W in the form of a Čech cocycle for the cover (see this later).

Theorem: There is a one to one correspondence between $\mathbb{G}_a$ bundles over W and extensions $\hat{\delta} = \delta + p(x_1, y_1, x_2, y_2, z)\frac{\partial}{\partial u}$ of δ. The total space X is affine iff $\hat{\delta}(q) = 1$ for some $q \in \mathbb{C}[x_1, y_1, x_2, y_2, z, u]$, and in this case $X = \mathbf{Spec} \ker \hat{\delta} \cong \mathcal{V}(q)$.

Example: The kernel of the extension $\hat{\delta} = \delta - \frac{\partial}{\partial u}$ contains $x_1, y_1,\ x_2 + ux_1,$ $y_2 + uy_1, z + u(1 + x_1y_2 - x_2y_1)$. Adjoining u to the ring generated by these 5 yields $\mathbb{C}[x_1, y_1, x_2, y_2, z, u]$, so they generate the $\ker(\hat{\delta})$ and we recover the total space $\mathbb{C}^5$ in fact the original action.

Can we construct an example where the total space is affine but not $\mathbb{C}^5$?

Not if $p \in \mathbb{C}[x_1, y_1, x_2, y_2]$.

Example The extensions $\hat{\delta}_n = \delta + x_2^n \frac{\partial}{\partial u}$ correspond to $\mathbb{G}_a$ bundles over W with affine total spaces. Recall that $c_3 = x_1 y_2 - x_2 y_1$. The regular function

$$s_n = \frac{z(1 - c_3^n)}{1 + c_3} + \frac{x_2(x_1 y_2 - x_2 y_1)^n - y_1^n x_2^{n+1}}{x_1} + (n+1) y_1^n u$$

satisfies $\hat{\delta}_n(s_n) = 1$. But s_n is actually a variable of $\mathbb{C}[x_1, y_1, x_2, y_2, z, u]$ (a fact that took me two years to prove) so ker $\hat{\delta} \cong \mathbb{C}^5$, although as a bundle it is not the same as the original.

Can we determine whether the total space is affine without having to solve the PDE $\hat{\delta}(q) = 1$?

Remark: Because W is not affine, the trivial bundle does not have an affine total space.

For any extended derivation $\hat{\delta}$ the entries of the cocycle $(s_3 - s_2, s_3 - s_1, s_2 - s_1)$ for the associated bundle are calculated as

$$\begin{aligned} s_2 - s_1 &= \exp(-\frac{x_2}{x_1})\hat{\delta}(u) - \exp(-\frac{y_2}{y_1})\hat{\delta}(u) \\ &= \frac{h}{c_1^k c_2^i} \end{aligned}$$

where $h \in \mathcal{O}(W)$ and i, k are positive integers (similarly for $s_3 - s_2, s_3 - s_1$ but these don't actually matter).

Theorem Let Z be the total space for a nontrivial $\mathbb{G}_a$ bundle over W and let $\frac{h}{c_1^k c_2^i}$ be as above, with the further hypothesis that $h \notin (c_1, c_2)\mathcal{O}(Y)$. Then Z is affine if and only h restricts to a nonzero constant on $H = Y - W$, i.e. on the zero locus of $(c_1, c_2, c_3 + 1)$ in Y. The bundle is trivial if and only if $\hat{\delta}(u) \in im(\delta)$ (which can be checked algorithmically).

The affineness criterion is very similar to why D_n is an affine total space for a bundle over a non-separated scheme. The cocycle (gluing data) $\frac{2}{x^n}$ has a pole over what prevents the base space from being a variety. Similarly, the condition on h gives a pole over what prevents the base space W from being affine (essentially the codimension 2 condition on $Y - W$).

Example The extended derivation $\delta + zp(x_1, y_1, x_2, y_2, z)\frac{\partial}{\partial u}$does not yield an affine total space by this theorem. A calculation shows that for $\frac{h}{c_1^k c_2^i}$,the h vanishes identically on the zero locus of $(c_1, c_2, c_3 + 1))$. However, again by the theorem, the extended derivation $\delta + zp(x_1, y_1, x_2, y_2, z) + q(x_1, y_1, x_2, y_2, z)\frac{\partial}{\partial u}$ will yield an affine total space provided $\delta + q(x_1, y_1, x_2, y_2, z)\frac{\partial}{\partial u}$ does. So this is where to look for cancellation counterexamples.

Maybe in two years???????????