Elementary Extensions

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First-order structures (models)

Definition

1. *First-order structure is a set with sets of functions, relations and constants:* $\mathfrak{M} = (M, \{f_i\}_{i \in I}, \{R_j\}_{j \in J}, \{c_k\}_{k \in K})$
2. $\mathfrak{M}$ *is a substructure of* $\mathfrak{N}$ *if the functions and relations of* $\mathfrak{M}$ *are restrictions of the corresponding functions and relations on* $\mathfrak{N}$*, and* $\mathfrak{M}$ *and* $\mathfrak{N}$ *share the same constants.*
3. *For* $\bar{a} \in M^n$*, the type of* $\bar{a}$*,* $\mathrm{tp}^{\mathfrak{M}}(\bar{a})$ *is* $\{\varphi(\bar{x}) : \mathfrak{M} \models \varphi(\bar{a})\}$.

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Elementary extensions

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An extension $\mathfrak{M} \subseteq \mathfrak{N}$ *is elementary if for all* $\bar{a} \in \bigcup_{n\in\omega} M^n$,
$\mathrm{tp}^{\mathfrak{M}}(\bar{a}) = \mathrm{tp}^{\mathfrak{N}}(\bar{a})$.

Example

Let $\mathfrak{M} = (\omega, <)$ *and* $\mathfrak{N} = (\{-1\} \cup \omega, <)$. *Then the extension* $\mathfrak{M} \subseteq \mathfrak{N}$ *is not elementary:*

$$\mathrm{tp}^{\mathfrak{M}}(0) \neq \mathrm{tp}^{\mathfrak{N}}(\bar{0}).$$

In fact, for all $\bar{a} \in \bigcup_{n\in\omega} \omega^n$

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Elementary extensions exist

Theorem

Every infinite model has a proper elementary extension.

Proof.

Let $\mathfrak{M}$ be infinite, and let $T = \mathrm{Th}(\mathfrak{M}, a)_{a \in M}$. Let c be a new constant, and let $T' = T \cup \{c \neq a : a \in M\}$. By the compactness theorem T' has a model $\mathfrak{N}$. Since $\mathfrak{N} \models T$, $\mathfrak{N}$ is an elementary extension of $\mathfrak{M}$. □

Corollary

Every infinite model $\mathfrak{M}$ has elementary extensions of arbitrary large cardinalities $> |M|$.

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Elementary submodels exist

Theorem (Löwenheim, Skolem)

Let $\mathfrak{M}$ be a model with at most countably many functions and constants. Then $\mathfrak{M}$ has an elementary countable submodel.

Example

$(\mathbb{R}, +, \times, 0, 1)$ *has* $2^{\aleph_0}$ *countable real closed subfields.*

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Rings and field extensions

None of these extensions are elementary:

$$(\mathbb{N},+,\times,0,1)\subseteq(\mathbb{Z},+,\times,0,1)\subseteq(\mathbb{Q},+,\times,0,1)\subseteq$$

$$\subseteq(\mathbb{R},+,\times,0,1)\subseteq(\mathbb{C},+,\times,0,1).$$

Theorem

$\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ have $2^{\aleph_0}$ countable elementary extensions.

Theorem

If $\mathfrak{R}$ and $\mathfrak{R}'$ are real closed fields and $\mathfrak{R}\subseteq\mathfrak{R}'$, then $\mathfrak{R}\prec\mathfrak{R}'$

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An application: automorphisms of $\mathbb{C}$

Proposition

There is a nontrivial $\alpha \in \mathrm{Aut}(\mathbb{C}, +, \times, 0, 1)$, *such that* α *is not* $z \mapsto \bar{z}$.

Proof.

Let $\mathbb{R} \prec \mathbb{R}^*$ be a proper elementary extension and let $\mathbb{C}^*$ be the algebraic closure of $\mathbb{R}^*$. $\mathbb{C}^* = \{a + bi : a,\ b \in \mathbb{R}^*\}$. Let β be the conjugation in $\mathbb{C}^*$. We can assume that $|\mathbb{R}| = |\mathbb{R}^*|$; hence $\mathbb{C}^* \cong \mathbb{C}$. Let $F : \mathbb{C} \cong \mathbb{C}^*$ be an isomorphism. Let $\alpha = F^{-1} \circ \beta \circ F$.

$$\mathrm{fix}(\alpha) \cong \mathbb{R}^* \ncong \mathbb{R}.$$

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An application: minimality of $(\omega, <)$

Theorem

If $X \subseteq \omega$ is definable in $(\omega, <)$, then X is either finite or cofinite.

Proof.

Suppose $X = \{n : (\omega, <) \models \varphi(n)\}$ and X is neither finite nor cofinite:

$$(\omega, <) \models \forall x \exists y \exists z \, [x, y > z \wedge \varphi(y) \wedge \neg\varphi(z)]$$

Since $(\omega, <) \prec (\omega + (\omega * + \omega), <)$, the same statement is true in the extension. Hence there are nonstandard a, b such that $(\omega + (\omega * + \omega), <) \models \varphi(a) \wedge \neg\varphi(b)$. There is an $\alpha \in \mathrm{Aut}(\omega + (\omega * + \omega), <)$ such that $\alpha(a) = b$. Contradiction. $\square$

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Models with automorphisms

Theorem (Ehrenfeucht, Mostowski)

Every infinite model has an elementary extension with a nontrivial automorphism.

Theorem

If $\mathfrak{M}$ is infinite and $\mathrm{tp}^{\mathfrak{M}}(\bar{a}) = \mathrm{tp}^{\mathfrak{M}}(\bar{b})$, then there is an $\mathfrak{N}$ such that $\mathfrak{M} \prec \mathfrak{N}$ and there is an $\alpha \in \mathrm{Aut}(\mathfrak{N})$ such that $\alpha(\bar{a}) = \alpha(\bar{b})$.

Theorem (Schmerl)

Let $(\mathfrak{A}, <, \dots)$ be a linearly ordered structure (or, equivalently, a left-orderable group). Then, there is $\mathfrak{M} = (M, +, \times, 0, 1)$ such that $\mathbb{N} \prec \mathfrak{M}$ and $\mathrm{Aut}(\mathfrak{A}, <, \dots) \cong \mathrm{Aut}(\mathfrak{M})$.

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