WA 4

[1] It is not possible to construct such a magic square. We give a proof by contradiction: we assume that

a	b	c
d	3	e
f	g	h

is a magic square with $\{a,b,c,d,e,f,g,h\} = \{1,2,4,5,6,7,8,9\}$ and we will deduce a contradiction.

First, we compute the sum of all the entries in the square:

$$a+b+c+d+e+f+g+h+3 = 1+2+3+4+5+6+7+8+9 = 45 \qquad (1)$$

Next, let s be the sum of the three entries in any row, column or diagonal of the square. Then, considering the square's three rows, we find:

$$a+b+c = d+3+e = f+g+h = s$$

As a consequence

$$3s = a+b+c+d+3+e+f+g+h \qquad (2)$$

From equations (1) and (2) we deduce that $3s = 45$, hence

$$s = 15 \qquad (3)$$

Since the sum of the three entries in the row, the column and the two diagonals through the center of the square is equal to s we find

$$d+3+e = 15 \; ; \; b+3+g = 15 \; ; \; a+3+h = 15 ; \; f+3+c = 15 \qquad (4)$$

As a consequence

$$d+e = 12 \; ; \; b+g = 12 \; ; \; a+h = 12 \; ; \; f+c = 12 \qquad (5)$$

Adding the four equations in (5) we obtain

$$a+b+c+d+e+f+g+h = 48 \qquad (6)$$

On the other hand, subtracting 3 from both sides of eq. (1) we get

$$a+b+c+d+e+f+g+h = 42 \qquad (7)$$

The contradiction between (6) and (7) proves that it is not possible to construct a magic square of the prescribed form.

$\boxed{2}$ The proposition is true.

<u>Proposition</u>: For each integer n, if n is odd, then $8 \mid (n^2-1)$.

<u>Proof</u>: Let n be an odd integer. We will prove that $8 \mid (n^2-1)$.

Since n is odd, there exists an integer $k \in \mathbb{Z}$ such that

$$n = 2k+1 \qquad (1)$$

It then follows that

$$\begin{aligned} n^2-1 &= (2k+1)^2-1 \\ &= 4k^2+4k+1-1 \\ &= 4k^2+4k \qquad (2) \end{aligned}$$

We now consider two cases: either k is even or k is odd.

In the case where k is even, then there exists $q \in \mathbb{Z}$ such that $k=2q$. Substituting this into equation (2) we find

$$\begin{aligned} n^2-1 &= 4(2q)^2+4(2q) \\ &= 8(2q^2)+8q \\ &= 8(2q^2+q) \qquad (3) \end{aligned}$$

Since $2q^2+q \in \mathbb{Z}$, equation (3) shows that $8 \mid (n^2-1)$ when k is even.

In the case where k is even, then there exists $r \in \mathbb{Z}$ such that $k = 2r+1$. Substituting into equation (2) we find

$$\begin{aligned} n^2-1 &= 4(2r+1)^2 + 4(2r+1) \\ &= 4(4r^2+4r+1) + 4(2r+1) \\ &= 16r^2 + 24r + 8 \\ &= 8(2r^2+3r+1) \qquad (4) \end{aligned}$$

As $2r^3+3r+1 \in \mathbb{Z}$, equation (4) shows that $8 \mid (n^2-1)$ when k is odd.

So, in both cases, we have proven that $8 \mid (n^2-1)$. ⊠

$\boxed{3}$ <u>Proposition:</u> Let a be a positive real number. For each real number x,

$$|x| > a \text{ if and only if } x > a \text{ or } x < -a.$$

[We give two proofs]

<u>Proof</u>: Using the logical equivalence

(version 1)

$$P \longleftrightarrow Q \equiv \neg P \longleftrightarrow \neg Q$$

the proposition is equivalent to the statement: For each real number x, $|x| \leq a$ if and only if $x \leq a$ and $x \geq -a$, which is true by Exercise 11(b) of Section 3.4 ⊠.

<u>Proof</u> : To prove this biconditional statement, we have (Version 2) to prove the following two conditional statements:

(a) for every $x \in \mathbb{R}$, if $|x| > a$ then $x > a$ or $x < -a$

(b) for every $x \in \mathbb{R}$, if $x > a$ or $x < -a$ then $|x| > a$.

Let a be a positive real number and $x \in \mathbb{R}$. We begin with the proof of (a). We assume that $|x| > a$. We consider two cases: $x \geqslant 0$ and $x < 0$. In the case where $x \geqslant 0$, then $|x| = x$ and therefore ~~$|x|>a$~~ we can conclude from $|x| > a$ that $x > a$. On the other hand, ~~if~~ in the case where $x < 0$, then $|x| = -x$. Therefore $|x| > a$ implies that $-x > a$. Multiplying this inequality by (-1) we can conclude that $x < -a$. This proves the conditional statement (a).

We now prove (b). We assume that $x > a$ or $x < -a$. We consider the two cases separately. In case $x > a$, then $x > 0$ since a is positive. Therefore $|x| = x$, and so $x > a$ gives us that $|x| > a$. In case $x < -a$, then $x < 0$ and so $|x| = -x$. Multiplying $x < -a$ by (-1) we obtain that $-x > a$ which can be written as $|x| > a$. We have shown that $|x| > a$ in both cases. This proves (b) and finishes the proof of the proposition. $\boxtimes$