WA 5

1 **Proposition:** For each natural number n,

$$1+5+9+\cdots+(4n-3)=n(2n-1)$$

Proof: We will prove this by mathematical induction. For each natural number n, we let $P(n)$ be

$$1+5+9+\cdots+(4n-3)=n(2n-1).$$

We first prove that $P(1)$ is true. Notice that for $n=1$ the left-hand side of $P(n)$ is equal to 1, while the right-hand side is equal to $(1)(2(1)-1)=1$. This shows that $P(1)$ is true.

For the inductive step, we prove that for each $k\in\mathbb{N}$, if $P(k)$ is true, then $P(k+1)$ is true. Let $k\in\mathbb{N}$ and assume that $P(k)$ is true. That is, assume

$$1+5+9+\cdots+(4k-3)=k(2k-1) \qquad (1)$$

Our goal is to prove $P(k+1)$ is true: it must be proved that

$$\begin{aligned}1+5+9+\cdots+(4k-3)+(4(k+1)-3) &= (k+1)(2(k+1)-1)\\ &= (k+1)(2k+1)\\ &= 2k^2+3k+1 \qquad (2)\end{aligned}$$

To do this, we add $4(k+1)-3$ to both sides of eqn. (1) and algebraically rewrite the resulting equation

$$\begin{aligned}1+5+9+\cdots+4(k-3)+4(k+1)-3 &= k(2k-1)+4(k+1)-3\\ &= k(2k-1)+4k+1\\ &= 2k^2-k+4k+1\\ &= 2k^2+3k+1 \qquad (3)\end{aligned}$$

Comparing equations (2) and (3) we see that if $P(k)$ is true, then so is $P(k+1)$. This establishes the inductive step and completes the proof of the proposition $\boxtimes$

$\boxed{2}$ <u>Proposition</u>: For each natural number n, $3 \mid (4^n - 1)$

<u>Proof</u>: We will prove this by mathematical induction. For every $n \in \mathbb{N}$ let $P(n)$ be

$$3 \mid (4^n - 1).$$

Since $4^1 - 1 = 3$ and $3 \mid 3$, we see that $P(1)$ is true.

For the inductive step, let $k \in \mathbb{N}$ and assume that $P(k)$ is true. That is, we assume that

$$3 \mid (4^k - 1) \qquad (1)$$

Our goal now is to prove that $P(k+1)$ is true; that is, we want to prove that

$$3 \mid (4^{k+1} - 1) \qquad (2)$$

By (1) we know that there exists $m \in \mathbb{Z}$ such that $4^k - 1 = 3m$, which is equivalent to

$$4^k = 1 + 3m \qquad (3)$$

Multiplying both sides of (3) by 4 we obtain

$$\begin{aligned} 4^{k+1} &= 4(1+3m) \\ &= 4 + 12m \\ &= 1 + 3 + 12m \\ &= 1 + 3(1+4m) \qquad (4) \end{aligned}$$

Since $1+4m \in \mathbb{Z}$, it follows from equation (4) that $3 \mid (4^{k+1}-1)$.
We have proven that if $P(k)$ is true, then so is $P(k+1)$.
This establishes the inductive step and completes the proof of the proposition. ☒

$\boxed{3}$ 15. (c) i. $[-3,7] \cap (5,9] = (5,7]$

ii. $[-3,7] \cup (5,9] = [-3,9]$

iii. $[-3,7] - (5,9] = [-3,5]$

(d) $\{x \in \mathbb{R} \mid |x| \leq 0.01\} = [-0.01, 0.01]$

(e) $\{x \in \mathbb{R} \mid |x| > 2\} = (-\infty,-2) \cup (2,\infty)$

16. (a) $[2,5] \cap [-1,\infty) = [2,5]$

$[2,5] \cup [-1,\infty) = [-1,\infty)$

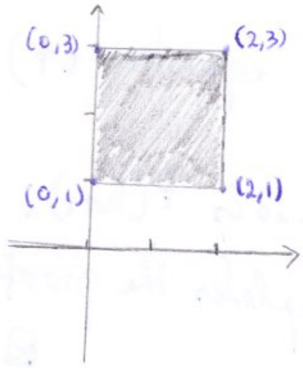
(b) $[2,5] \cap [3.4,\infty) = [3.4,5]$

$[2,5] \cup [3.4,\infty) = [2,\infty)$

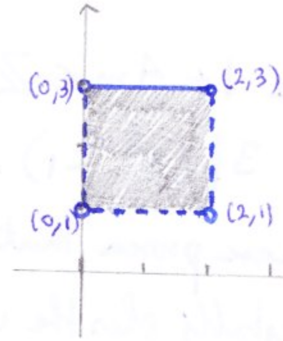
(c) $[2,5] \cap [7,\infty) = \emptyset$

union is $[2,5] \cup [7,\infty)$

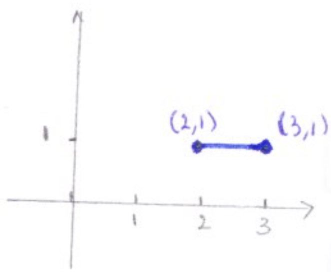
4 (a)



(b)



(c)



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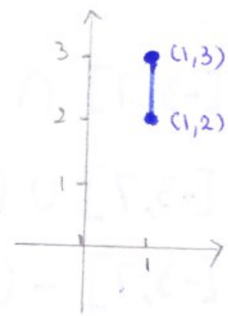

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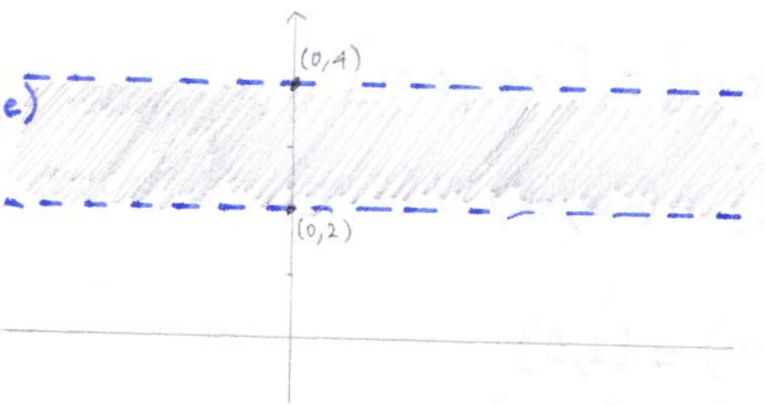

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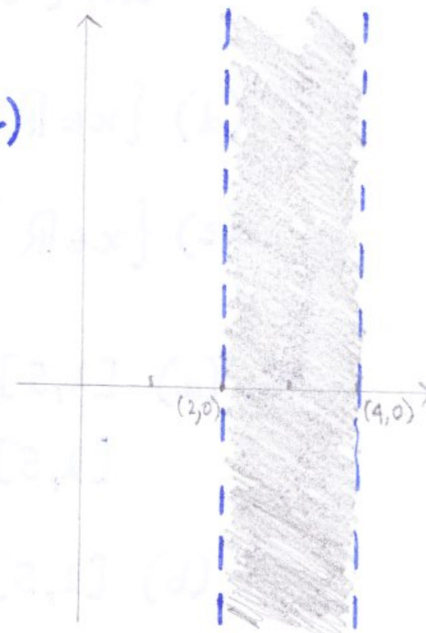
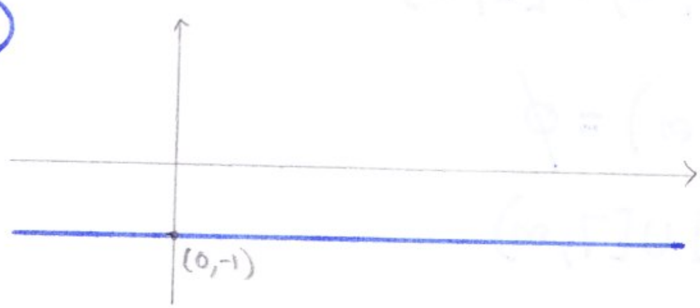


(g)



(h)