P Set 4

1 Let $\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n$ be the rows of A

Then $\det(3A) = \det(3\underline{a}_1, 3\underline{a}_2, \dots, 3\underline{a}_n) \underset{\uparrow}{=} 3^n \det(\underline{a}_1, \dots, \underline{a}_n) = 3^n \det(A)$

det is linear in each row

2 (a) $\det B = 3.5.7 \det(A)$ because det is linear in each column

(b) let $\underline{v}_1, \underline{v}_2, \underline{v}_3$ be the columns of A. Then

$$\det(B) = \det(3\underline{v}_1,\ 4\underline{v}_2 + 5\underline{v}_1,\ 5\underline{v}_3)$$

$$= 3.5 \det(\underline{v}_1,\ 4\underline{v}_2 + 5\underline{v}_1,\ \underline{v}_3) \quad \text{by linearity in entry 1 \& 3}$$

$$= 3.5\left(4\det(\underline{v}_1, \underline{v}_2, \underline{v}_3) + 5\underbrace{\det(\underline{v}_1, \underline{v}_1, \underline{v}_3)}_{\substack{\parallel \\ 0}}\right) \quad \text{by linearity in entry 2}$$

because det is antisymmetric

$$= 3.4.5 \det(\underline{v}_1, \underline{v}_2, \underline{v}_3) = 3.4.5 \det(A)$$

3

$$\det\begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix} \overset{\substack{-R_1+R_2 \\ -R_1+R_3}}{=} \det\begin{pmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{pmatrix}$$

det is linear in row 2 and row 3

$$\downarrow$$

$$= (b-a)(c-a)\det\begin{pmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{pmatrix} \overset{-R_2+R_3}{=} (b-a)(c-a)\det\begin{pmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{pmatrix}$$

det of upper triangular

$$\downarrow$$

$$= (b-a)(c-a)(c-b)$$

$\boxed{4}$ $n=1.$ $A+tI_n = (a_0+t)$ $\det(A) = t + a_0$

$n=2$ $A+tI_n = \begin{pmatrix} t & a_0 \\ -1 & a_1+t \end{pmatrix}$ $\det(A) = t(a_1+t) + a_0$

$= t^2 + a_1 t + a_0$

claim: $\det(A+tI_n) = t^n + a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \ldots + a_1 t + a_0$

proof of claim by induction on n

basic case: $(n=1)$ see above

induction step: suppose the claim holds for $n=k$. We prove that it then holds for $n=k+1$.

$$\det(A+tI_{k+1}) = \det\begin{pmatrix} t & 0 & 0 & \cdots & 0 & a_0 \\ -1 & t & 0 & \cdots & 0 & a_1 \\ 0 & -1 & t & \cdots & 0 & a_2 \\ \vdots & 0 & -1 & & & \\ & \vdots & & & t & a_{k-1} \\ 0 & 0 & & & -1 & t+a_k \end{pmatrix}$$

expand along first row

$$= t\cdot\det\begin{pmatrix} t & 0 & 0 & \cdots & 0 & a_1 \\ -1 & t & 0 & \cdots & 0 & a_2 \\ 0 & \ddots & & & \vdots & \vdots \\ \vdots & & & & 0 & \vdots \\ \vdots & & & \ddots & t & a_{k-1} \\ 0 & \cdots & & 0 & -1 & t+a_k \end{pmatrix} + (-1)^k a_0 \det\begin{pmatrix} -1 & t & 0 & \cdots & & 0 \\ 0 & -1 & t & 0 & \cdots & 0 \\ 0 & 0 & \ddots & & & \\ \vdots & & & \ddots & & \\ & & & & \ddots & t \\ 0 & 0 & \cdots & & 0 & -1 \end{pmatrix}$$

induction hypothesis + det of upper triangular

$$= t\cdot\left(t^k + a_k t^{k-1} + a_{k-1}t^{k-2} + \ldots + a_1\right) + a_0(-1)^k(-1)^k$$

$$= t^{k+1} + a_k t^k + a_{k-1}t^{k-1} + \ldots + a_1 t + a_0 \quad . \boxtimes$$