PSet 6 (Corrected 5/13/18)

[1] A square matrix is invertible if and only if its determinant is nonzero.

Set $A = \begin{pmatrix} \alpha & \alpha & \alpha \\ \alpha & 1 & 1 \\ \alpha & 1 & \alpha \end{pmatrix}$.

$$\det A = \alpha \det \begin{pmatrix} 1 & 1 & 1 \\ \alpha & 1 & 1 \\ \alpha & 1 & \alpha \end{pmatrix}$$

$$= \alpha \det \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1-\alpha & 1-\alpha \\ 0 & 1-\alpha & 0 \end{pmatrix} \qquad [-\alpha R_1 + R_2 \to R_2]$$

$$= \alpha \det \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1-\alpha & 1-\alpha \\ 0 & 0 & \alpha-1 \end{pmatrix} \qquad [-R_2 + R_3 \to R_3]$$

$$= \alpha(1-\alpha)(\alpha-1)$$

Consequently, A is invertible $\Leftrightarrow \alpha \notin \{0,1\}$

We compute the inverse when $\alpha \notin \{0,1\}$

$$\left(\begin{array}{ccc|ccc} \alpha & \alpha & \alpha & 1 & 0 & 0 \\ \alpha & 1 & 1 & 0 & 1 & 0 \\ \alpha & 1 & \alpha & 0 & 0 & 1 \end{array}\right) \overset{\frac{1}{\alpha}R_1}{\sim} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & \alpha^{-1} & 0 & 0 \\ \alpha & 1 & 1 & 0 & 1 & 0 \\ \alpha & 1 & \alpha & 0 & 0 & 1 \end{array}\right)$$

$$\underset{-\alpha R_1 + R_3}{\overset{-\alpha R_1 + R_2}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & \alpha^{-1} & 0 & 0 \\ 0 & 1-\alpha & 1-\alpha & -1 & 1 & 0 \\ 0 & 1-\alpha & 0 & -1 & 0 & 1 \end{array}\right) \overset{\frac{1}{1-\alpha}R_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & \alpha^{-1} & 0 & 0 \\ 0 & 1 & 1 & (\alpha-1)^{-1} & (1-\alpha)^{-1} & 0 \\ 0 & 1-\alpha & 0 & -1 & 0 & 1 \end{array}\right)$$

$$\overset{(\alpha-1)R_2+R_3}{\sim}\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & \alpha^{-1} & 0 & 0 \\ 0 & 1 & 1 & (\alpha-1)^{-1} & (1-\alpha)^{-1} & 0 \\ 0 & 0 & \alpha-1 & 0 & -1 & 1 \end{array}\right)$$

$$\overset{\frac{1}{\alpha-1}R_3}{\sim}\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & \alpha^{-1} & 0 & 0 \\ 0 & 1 & 1 & (\alpha-1)^{-1} & (1-\alpha)^{-1} & 0 \\ 0 & 0 & 1 & 0 & (1-\alpha)^{-1} & (\alpha-1)^{-1} \end{array}\right)$$

$$\overset{-R_2+R_1}{\sim}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \alpha^{-1}+(1-\alpha)^{-1} & (\alpha-1)^{-1} & 0 \\ 0 & 1 & 1 & (\alpha-1)^{-1} & (1-\alpha)^{-1} & 0 \\ 0 & 0 & 1 & 0 & (1-\alpha)^{-1} & (\alpha-1)^{-1} \end{array}\right)$$

$$\overset{-R_3+R_2}{\sim}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \alpha^{-1}+(1-\alpha)^{-1} & (\alpha-1)^{-1} & 0 \\ 0 & 1 & 0 & (\alpha-1)^{-1} & 0 & (1-\alpha)^{-1} \\ 0 & 0 & 1 & 0 & (1-\alpha)^{-1} & (\alpha-1)^{-1} \end{array}\right)$$

so $$A^{-1} = \begin{pmatrix} \frac{1}{\alpha}+\frac{1}{1-\alpha} & \frac{1}{\alpha-1} & 0 \\ \frac{1}{\alpha-1} & 0 & \frac{1}{1-\alpha} \\ 0 & \frac{1}{1-\alpha} & \frac{1}{\alpha-1} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\alpha(1-\alpha)} & \frac{1}{\alpha-1} & 0 \\ \frac{1}{\alpha-1} & 0 & \frac{1}{1-\alpha} \\ 0 & \frac{1}{1-\alpha} & \cancel{0}\ \frac{1}{\alpha-1} \end{pmatrix}$$

$\boxed{2}$ (a) $\det A = s \det \begin{pmatrix} -1 & s & -1 \\ 0 & -1 & s \\ s & 0 & 0 \end{pmatrix}$

$$= s \det \begin{pmatrix} -1 & s & -1 \\ 0 & -1 & s \\ 0 & s^2 & -s \end{pmatrix} \quad [sR_1 + R_3]$$

$$= s \det \begin{pmatrix} -1 & s & -1 \\ 0 & -1 & s \\ 0 & 0 & s^3 - s \end{pmatrix} \quad [s^2R_2 + R_3]$$

$$= s(s^3 - s)$$

$$= s^2(s^2-1) = s^2(s+1)(s-1)$$

~~so~~ A invertible $\Leftrightarrow \det A \neq 0$

$\Leftrightarrow s \notin \{0,-1,1\} \Leftrightarrow s \in \mathbb{R} \setminus \{0,-1,1\}$

(b) $\det B = \begin{pmatrix} 1 & 1 & s \\ 0 & 1-s & 0 \\ 0 & 0 & 1-s^2 \end{pmatrix} \quad [-sR_1 + R_3]$

$$= (1-s)(1-s^2) = (1-s)(1-s)(1+s)$$

B invertible $\Leftrightarrow s \notin \{-1,1\} \Leftrightarrow s \in \mathbb{R} \setminus \{-1,1\}$

(c) When $s \in \mathbb{R} \setminus \{-1,1,0\}$ then both A and B are invertible and therefore $A.B$ is invertible, which means that $\operatorname{rank}(A.B) = 3$

When $s \in \{-1,1,0\}$ then $\det(A.B) = \det(A).\det(B) = 0$ and so $\operatorname{rank}(A.B) \leq$ ~~3~~ 2

We analyze the remaining values for s:

(i) $s=0$:

$$A.B = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \boxed{\begin{matrix} 0 & -1 \\ -1 & 0 \end{matrix}} \\ 0 & \\ 0 & \begin{matrix} 0 & 0 \end{matrix} \end{pmatrix}$$

since the boxed matrix has nonzero determinant, $\operatorname{rank}(A.B) = 2$

(ii) $s=-1$

$$A.B = \begin{pmatrix} 1 & -1 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \boxed{\begin{matrix} 2 & 0 \\ 1 & -1 \end{matrix}} & \begin{matrix} -2 \\ 1 \end{matrix} \\ \begin{matrix} 1 & 1 \end{matrix} & -1 \end{pmatrix}$$

the boxed minor is nonzero, and therefore $\operatorname{rank}(A.B) = 2$

(iii) $s=1$

$$A.B = \begin{pmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & -2 & -2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

since all rows are multiples of the third row, $A.B$ contains 1 linearly independent row and so $\operatorname{rank}(A.B) = 1$

$\boxed{3}$ (a) Let B be the matrix with columns b_1, b_2 and b_3. Then B is a basis of $\mathbb{R}^3$ if and only if $\det B \neq 0$. Now, $\det B = \det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = 1$ and so b_1, b_2, b_3 is a basis of $\mathbb{R}^3$.

Similarly, let C be the matrix with columns c_1 and c_2.
Now $\det C = \det \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} = 4-3=1$, and so c_1, c_2 is a basis of $\mathbb{R}^2$.

(b)
$$\varphi(b_1) = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \overset{*}{=} 2c_1 - c_2$$

$$\varphi(b_2) = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \overset{*}{=} -6c_1 + 4c_2$$

$$\varphi(b_3) = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix} \overset{*}{=} \cancel{-15c_1} -15c_1 + 10c_2$$

(technically, the equalities (*) involve solving a system of lin. equations)

consequently, the matrix of φ with respect to the bases in part (a) is
$$\begin{pmatrix} 2 & -6 & -15 \\ -1 & 4 & 10 \end{pmatrix}$$

[AS AN ALTERNATIVE, ONE CAN COMPUTE THIS MATRIX AS
$C^{-1}\begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & 1 \end{pmatrix} B$. DO YOU SEE WHY?]

$\boxed{4}$

$$\varphi_1(A_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = 2A_1$$

$$\varphi_1(A_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A_2 + A_3$$

$$\varphi_1(A_3) = A_2 + A_3$$

$$\varphi_1(A_4) = 2A_4$$

so the matrix of φ_1 with respect to the given matrix is

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

$$\varphi_2(A_1) = A_1 \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = A_1 + A_2$$

$$\varphi_2(A_2) = A_2 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = A_2$$

$$\varphi_2(A_3) = A_3 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = A_3 + A_4$$

$$\varphi_2(A_4) = A_4 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = A_4$$

and the matrix of φ_2 with respect to the given basis is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$