Pset 8 (Corrected 5/20/18)

1 • $A = \begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix}$

$$\chi_A(t) = \det(A - t\,Id) \overset{\det}{=} \begin{pmatrix} 4-t & -5 \\ 2 & -3-t \end{pmatrix}$$

$$= (4-t)(-3-t) + 10$$

$$= t^2 - t - 2$$

$$\boxed{\chi_A(t) = (t-2)(t+1)}$$

The eigenvalues of A are $\boxed{\lambda_1 = 2}$ and $\boxed{\lambda_2 = -1}$.

Eigenvectors for $\lambda_1 = 2$:

$$\begin{pmatrix} 4-2 & -5 \\ 2 & -3-2 \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 2 & -5 \end{pmatrix} \underset{-R_1+R_2}{\sim} \begin{pmatrix} 2 & -5 \\ 0 & 0 \end{pmatrix} \qquad 2x - 5y = 0 \text{ or } x = \frac{5y}{2}$$

$$\text{Ker}(A - 2\,Id) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2 \;\middle|\; x = \frac{5y}{2} \right\}$$

$$= \left\{ y \begin{pmatrix} 5/2 \\ 1 \end{pmatrix} \;\middle|\; y \in \mathbb{C} \right\}$$

$$= \left\langle \begin{pmatrix} 5/2 \\ 1 \end{pmatrix} \right\rangle$$

Eigenvectors for $\lambda_2 = -1$

$$\begin{pmatrix} 4-(-1) & -5 \\ 2 & -3-(-1) \end{pmatrix} = \begin{pmatrix} 5 & -5 \\ 2 & -2 \end{pmatrix} \underset{-\frac{2}{5}R_1+R_2}{\sim} \begin{pmatrix} 5 & -5 \\ 0 & 0 \end{pmatrix}.$$

$$\text{Ker}(A - (-1)\,Id) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2 \;\middle|\; x = y \right\}$$

$$= \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle$$

- $A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$

$$\chi_A(t) = \det\begin{pmatrix} 2-t & 1 \\ -1 & 4-t \end{pmatrix} = (2-t)(4-t) + 1$$

$$= t^2 - 6t + 9 = \boxed{(t-3)^2}$$

A has only one eigenvalue: $\boxed{\lambda = 3}$

Eigenvectors for $\lambda = 3$:

$$A - 3\,\mathrm{Id} = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$$

so $\mathrm{Ker}(A - 3\,\mathrm{Id}) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2 \;\middle|\; x = y \right\} = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle$

- $A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$

$$\chi_A(t) = \det\begin{pmatrix} 1-t & 3 & 3 \\ -3 & -5-t & -3 \\ 3 & 3 & 1-t \end{pmatrix} \underset{R_2+R_3}{=} \det\begin{pmatrix} 1-t & 3 & 3 \\ -3 & -5-t & -3 \\ 0 & -2-t & -2-t \end{pmatrix}$$

$$\underset{-C_3+C_2}{=} \det\begin{pmatrix} 1-t & 0 & 3 \\ -3 & -2-t & -3 \\ 0 & 0 & -2-t \end{pmatrix} = (-2-t)^2(1-t) = \boxed{(t+2)^2(1-t)}$$

The eigenvalues are $\boxed{\lambda_1 = -2}$ and $\boxed{\lambda_2 = 1}$

Eigenvectors for $\lambda_1 = -2$:

$$A + 2\,\mathrm{Id} = \begin{pmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{pmatrix} \underset{\substack{R_1+R_2 \\ -R_1+R_2}}{\sim} \begin{pmatrix} 3 & 3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathrm{Ker}(A + 2\,\mathrm{Id}) = \left\{ \begin{pmatrix} -(y+z) \\ y \\ z \end{pmatrix} : y, z \in \mathbb{C} \right\}$$

$$= \left\{ y \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} : y, z \in \mathbb{C} \right\}$$

$$= \left\langle \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

Eigenvectors for $\lambda_2 = 1$:

$$A - \mathrm{Id} = \begin{pmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{pmatrix} \underset{R_1 \leftrightarrow R_3}{\sim} \begin{pmatrix} 3 & 3 & 0 \\ -3 & -6 & -3 \\ 0 & 3 & 3 \end{pmatrix} \underset{R_1+R_2}{\sim} \begin{pmatrix} 3 & 3 & 0 \\ 0 & -3 & -3 \\ 0 & 3 & 3 \end{pmatrix}$$

$$\underset{R_2+R_3}{\sim} \begin{pmatrix} 3 & 3 & 0 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{pmatrix} \underset{R_2+R_1}{\sim} \begin{pmatrix} 3 & 0 & -3 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathrm{Ker}(A - \mathrm{Id}) = \left\{ \begin{pmatrix} z \\ -z \\ z \end{pmatrix} : z \in \mathbb{C} \right\} = \left\langle \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\rangle$$

$\boxed{2}$ $A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$

$$\chi_A(t) = \det\begin{pmatrix} \cos\alpha - t & -\sin\alpha \\ \sin\alpha & \cos\alpha - t \end{pmatrix}$$

$$= (\cos\alpha - t)^2 + \sin^2\alpha$$

$$= \cos^2\alpha - 2t\cos\alpha + t^2 + \sin^2\alpha$$

$$\boxed{\chi_A(t) = t^2 - 2\cos\alpha\, t + 1}$$

Eigenvalues: $\chi_A(t) = 0 \iff t^2 - 2\cos\alpha\, t + 1 = 0$

$$t = \frac{2\cos\alpha \pm \sqrt{4\cos^2\alpha - 4}}{2}$$

$$= \frac{2\cos\alpha \pm 2\sqrt{\cos^2\alpha - 1}}{2}$$

$$= \cos\alpha \pm \sqrt{-\sin^2\alpha}$$

so eigenvalues are

$$\boxed{\begin{aligned} \lambda_1 &= \cos\alpha + i\sin\alpha = e^{i\alpha} \\ \lambda_2 &= \cos\alpha - i\sin\alpha = e^{-i\alpha} \end{aligned}}$$

Eigenvectors for $\lambda_1 = e^{i\alpha}$:

$$A - \lambda_1 \mathrm{Id} = \begin{pmatrix} \cos\alpha - e^{i\alpha} & -\sin\alpha \\ \sin\alpha & \cos\alpha - e^{i\alpha} \end{pmatrix} = \begin{pmatrix} -i\sin\alpha & -\sin\alpha \\ \sin\alpha & -i\sin\alpha \end{pmatrix}$$

$$\xrightarrow{-iR_1 + R_2} \begin{pmatrix} -i\sin\alpha & -\sin\alpha \\ 0 & 0 \end{pmatrix} \xrightarrow[\frac{1}{-\sin\alpha} \cdot R_1]{\text{IF } \alpha \notin \{k\pi \mid k \in \mathbb{Z}\}} \begin{pmatrix} +i & 1 \\ 0 & 0 \end{pmatrix} \quad ix = y$$

$$\mathrm{Ker}\,(A-\lambda_1 \mathrm{Id}) = \left\{ \begin{pmatrix} x \\ -ix \end{pmatrix} : x \in \mathbb{C} \right\} = \left\langle \begin{pmatrix} 1 \\ -i \end{pmatrix} \right\rangle$$

Eigenvectors for $\lambda_2 = e^{-i\alpha}$:

$$A - \lambda_2 \mathrm{Id} = \begin{pmatrix} i\sin\alpha & -\sin\alpha \\ \sin\alpha & i\sin\alpha \end{pmatrix} \underset{iR_1+R_2}{\sim} \begin{pmatrix} i\sin\alpha & -\sin\alpha \\ 0 & 0 \end{pmatrix}$$

IF $\alpha \notin \{k\pi \mid k \in \mathbb{Z}\}$

$$\underset{\frac{1}{\sin\alpha}R_1}{\sim} \begin{pmatrix} i & -1 \\ 0 & 0 \end{pmatrix} \qquad ix = y$$

$$\mathrm{Ker}\,(A-\lambda_2 \mathrm{Id}) = \left\langle \begin{pmatrix} 1 \\ i \end{pmatrix} \right\rangle$$

We still have to find the eigenvectors when $\alpha \in \{k\pi \mid k \in \mathbb{Z}\}$.

- If $\alpha \in \{2k\pi \mid k \in \mathbb{Z}\}$, then $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\lambda_1 = \lambda_2 = 1$.
  Every non zero vector in $\mathbb{C}^2$ is an eigenvector for $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ of eigenvalue 1.
- If $\alpha \in \{(2k+1)\pi \mid k \in \mathbb{Z}\}$, then $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\lambda_1 = \lambda_2 = -1$
  Every non zero vector in $\mathbb{C}^2$ is an eigenvector for $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ of eigenvalue -1.

$\boxed{3}$ a) False: $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ only has one eigenvalue

b) True: if v is an eigenvector, then so is kv for every $k \in \mathbb{C} \setminus \{0\}$

c) True: see problem $\boxed{2}$ above

d) False: the characteristic polynomial has at least one root and therefore the matrix has at least one eigenvector

e) True: similar matrices have the same ~~eigenvalues and~~ characteristic polynomial and therefore the same eigenvalues

f) False: let $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ and $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

then $S^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. set $B = SAS^{-1} = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$

Then $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an eigenvector for A, but not for B:

$$A \cdot e_2 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} = 3e_2$$

$$B \cdot e_2 = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \notin \langle e_2 \rangle$$

g) False: let $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. Then e_1 is an eigenvector and e_2 is an ~~eigenvector~~ eigenvector, but $e_1 + e_2$ is not an eigenvector:

$$A \cdot e_1 = 2e_1, \quad Ae_2 = 3e_2,$$

$$A \cdot (e_1 + e_2) = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \notin \langle e_1 + e_2 \rangle$$

h) True: let v and w be eigenvectors of A of eigenvalue λ; then $A \cdot (v+w) = A \cdot v + A \cdot w = \lambda v + \lambda w = \lambda(v+w)$

$\boxed{5}$ <u>lemma</u>: let A be a square matrix of size n.
The leading term of the characteristic polynomial $\chi_A(\lambda)$ of A is $(-1)^n\lambda^n$

<u>proof</u>: We prove this by induction on n.
If $n=1$ then $A=(a_{11})$ and $\chi_A(\lambda) = -\lambda + a_{11}$, which indeed has leading term $-\lambda$.
We assume the lemma holds for $n=k$. Let A be a $k\times k$-matrix. Since A has at least one eigenvector and similar matrices have the same characteristic polynomial, we may assume that A has the form

$$A = \begin{pmatrix} a_{11} & a_{12} \cdots a_{1,k+1} \\ 0 & \\ 0 & \boxed{\quad B \quad} \\ \vdots & \\ 0 & \end{pmatrix}$$

where B is a $(k\times k)$-matrix. Then developing $\det(A-\lambda\, Id)$ along the first column yields
$\chi_A(\lambda) = (a_{11}-\lambda)\,\chi_B(\lambda)$. By the induction hypothesis, the leading term of $\chi_B(\lambda)$ is $(-1)^k\lambda^k$, and consequently, the leading term of $\chi_A(\lambda)$ is $(-1)^{k+1}\lambda^{k+1}$, as desired. $\square$.

<u>Proposition</u>: let A be a square (complex) matrix of size n. Then $\det(A)$ is equal to the product of the eigenvalues of A.

<u>Proof</u>: By the lemma, the leading term of $(-1)^n\chi_A(\lambda)$ is λ^n.

Since $(-1)^n \chi_A(\lambda)$ has the same roots as $\chi_A(\lambda)$, this implies that $(-1)^n \chi_A(\lambda) = (\lambda - a_1)(\lambda - a_2) \dots (\lambda - a_n)$ where $a_1, a_2, \dots, a_n$ are the eigenvalues of A counted with their (algebraic) multiplicity.

This implies that $\chi_A(\lambda) = (-1)(\lambda - a_1)(-1)(\lambda - a_2) \dots (-1)(\lambda - a_n)$

$$= (a_1 - \lambda)(a_2 - \lambda) \dots (a_n - \lambda)$$

We have shown that $\det(A - \lambda \, \mathrm{Id}) = (a_1 - \lambda)(a_2 - \lambda) \dots (a_n - \lambda)$.

Setting $\lambda = 0$, this becomes $\det(A) = a_1 \cdot a_2 \cdot a_3 \dots a_n$, which is what we had to prove $\boxtimes$

$\boxed{4}$ All the matrices in this exercise are upper-triangular or lower-triangular.

Let A be such a matrix. Then $A - t\,\mathrm{Id}$ is also upper- or lower triangular, and so $\det(A - t\,\mathrm{Id})$ is the product of the diagonal entries of $A - t\,\mathrm{Id}$. So:

first matrix: $\chi_A(t) = (1-t)(2-t)(-2-t)(3-t)$

eigenvalues: $1, 2, -2, 3$

second matrix: $\chi_A(t) = (2-t)(\pi - t)(16-t)(54-t)$

eigenvalues: $2, \pi, 16, 54$

third matrix: $\chi_A(t) = (4-t)(3-t)(e-t)(1-t)$

eigenvalues: $4, 3, e, 1$

fourth matrix: $\chi_A(t) = (4-t)\,t^2\,(1-t)$

eigenvalues: $4, 0, 1$