quiz

$\boxed{1}$ (a) $\det\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix} = 2-1 \neq 0$, therefore f_1 and f_2 form a basis of $\mathbb{R}^2$

(b) $T(e_1) = T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix}$

$T(e_2) = T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix}$

$\Rightarrow \mathrm{Mat}_{EE}(T) = \begin{pmatrix}3 & 1\\ 1 & -1\end{pmatrix}$

(c) Solution 1:

$T(f_1) = T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}4\\0\end{pmatrix} = 8\begin{pmatrix}1\\1\end{pmatrix} + (-4)\begin{pmatrix}1\\2\end{pmatrix}$

solving a system of 2 linear equations in 2 unknowns (by inspection or any other technique)

$T(f_2) = T\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}5\\-1\end{pmatrix} = 11\begin{pmatrix}1\\1\end{pmatrix} + (-6)\begin{pmatrix}1\\2\end{pmatrix}$

$\Rightarrow \mathrm{Mat}_{FF}(T) = \begin{pmatrix}8 & 11\\ -4 & -6\end{pmatrix}$

Solution 2:

$\mathrm{Mat}_{FF}(T) = \mathrm{Mat}_{FE}(\mathrm{Id})\,\mathrm{Mat}_{EE}(T)\,\mathrm{Mat}_{EF}(\mathrm{Id})$

$\mathrm{Mat}_{EF}(\mathrm{Id}) = \begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}$

compute the inverse of $\mathrm{Mat}_{EF}(\mathrm{Id})$ OR use that $e_1 = 2f_1 - f_2$, $e_2 = -f_1 + f_2$

$\mathrm{Mat}_{FE}(\mathrm{Id}) = \begin{pmatrix}2 & -1\\ -1 & 1\end{pmatrix}$

$$\Rightarrow \mathrm{Mat}_{FF}(T) = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 3 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 11 \\ -4 & -6 \end{pmatrix}$$

$\boxed{2}$ $\cancel{\begin{pmatrix} -1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & \alpha \end{pmatrix} \overset{R_1+R.}{\sim}}$

$$\left(\begin{array}{ccc|ccc} -1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & \alpha & 0 & 0 & 1 \end{array}\right) \underset{R_1+R_3}{\overset{R_1+R_2}{\sim}} \left(\begin{array}{ccc|ccc} -1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 1 & 0 \\ 0 & 2 & 1+\alpha & 1 & 0 & 1 \end{array}\right)$$

$$\overset{-R_2+R_3}{\sim} \left(\begin{array}{ccc|ccc} -1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1+\alpha & 0 & -1 & 1 \end{array}\right)$$

the given matrix is invertible $\iff$ its echelon form has 3 pivots

$\iff 1+\alpha \neq 0$

$\iff \alpha \neq -1$

$\iff \boxed{\alpha \in \mathbb{R} \setminus \{-1\}}$

we now continue to compute the inverse matrix, assuming that $\alpha \neq -1$

$$\underset{\frac{1}{2}R_2}{\overset{(-1)R_1}{\sim}} \left(\begin{array}{ccc|ccc} 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1+\alpha & 0 & -1 & 1 \end{array}\right) \underset{R_2+R_1}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -1/2 & 1/2 & 0 \\ 0 & 1 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1+\alpha & 0 & -1 & 1 \end{array}\right)$$

$$\underset{\frac{1}{\alpha+1}R_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -1/2 & 1/2 & 0 \\ 0 & 1 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{1+\alpha} & \frac{1}{1+\alpha} \end{array}\right) \underset{R_3+R_1}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & \frac{1}{2}-\frac{1}{1+\alpha} & \frac{1}{1+\alpha} \\ 0 & 1 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{1+\alpha} & \frac{1}{1+\alpha} \end{array}\right)$$

$$\frac{1}{2} - \frac{1}{1+\alpha} = \frac{1+\alpha-2}{2(1+\alpha)} = \frac{\alpha-1}{2(1+\alpha)}$$

The inverse of the given matrix is

$$\begin{pmatrix} -1/2 & \frac{\alpha-1}{2(1+\alpha)} & \frac{1}{1+\alpha} \\ 1/2 & 1/2 & 0 \\ 0 & -\frac{1}{1+\alpha} & \frac{1}{1+\alpha} \end{pmatrix}$$

$\boxed{3}$ (a) Set $W = \{p \in V \mid p(2) = 0\}$

We have to prove that $U = W$

To show that $U \subseteq W$, it suffices to check that

$p_1(2) = (2)^2 - 4 = 0$

$p_2(2) = (2) - 2 = 0$

$p_3(2) = (2)^2 - 2(2) = 0$

To establish the other inclusion, we will show that

$\dim U = \dim W$.

First note that $\dim W = \dim V - 1$ since W is the ~~solu~~ kernel of a nontrivial linear map $V \longrightarrow \mathbb{R}$, namely the map $V \longrightarrow \mathbb{R}$, $p \longmapsto p(2)$.

(Put differently, W is the solution set of a single homogeneous linear equation on the coordinates a, b, c of an element $aX^2 + bX + c$ of V; namely the equation $4a + 2b + c = 0$).

Since $\dim V = 3$, this shows that $\dim W = 2$.

To prove that $\dim U = 2$, we will show that p_1, p_2 forms a basis of U. ~~We now set~~ Consider the matrix where the columns are the coordinates of p_1, p_2 and p_3 in the basis $x^2, x, 1$ of V, and row reduce it to echelon form

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ -4 & -2 & 0 \end{pmatrix} \underset{\sim}{4R_1 + R_3} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & -2 & 4 \end{pmatrix} \underset{\sim}{2R_2 + R_3} \begin{pmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

since the pivots are in columns 1 and 2, (p_1, p_2) is a basis of U, and so $\dim U = 2$ and $U = W$.

(b) We showed in part (a) that p_1, p_2 is a basis of U.

$\boxed{4}$ Observe that $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}$. The linearity of R tells us that $R\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$