Various interpretations of the root system(s) of a spherical variety

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The little Weyl group and the spherical roots. Let G be a complex connected reductive group and let B be a Borel subgroup of G. Recall that a normal irreducible complex algebraic variety X equipped with an action of G is called **spherical** if B has a dense orbit on it. We refer the reader to [13] or [14] for an introduction to spherical varieties. Throughout this paper, X will be a spherical G-variety and G/H will be its unique open G-orbit.

Two basic invariants of X are, using the notations of [14]:

- the subgroup $\Lambda(X)$ of the character group $X(B)$ of B consisting of the B-weights in the field $\mathbb{C}(X)$ of rational functions on X; and
- the so-called **valuation cone** $V(X)$, which is a convex polyhedral cone in $\Lambda^*_{\mathbb{Q}}(X) := \operatorname{Hom}_{\mathbb{Z}}(\Lambda(X), \mathbb{Q})$ ([12, Proposition 2.1] and [3, Corollaire 3.2]).

Note that these invariants only depend on the open G-orbit of X, that is, $\Lambda(X) = \Lambda(G/H)$ and $V(X) = V(G/H)$.

Another important birational invariant of X is its so-called little Weyl group W_X. It is defined by the following theorem, due to Brion [2, Theorem 3.5]. A completely different proof of (a generalization of) the theorem was given by Knop in [6, Theorem 7.4]. We combine the formulations of [5, Theorem 5.4] and [10, Theorem 1.1.4]. Let $T \subset B$ be a maximal torus, $W = N_G(T)/T$ the associated Weyl group and $N(\Lambda(X))$ the stabilizer in W of $\Lambda(X) \subset X(B) = X(T)$. We equip $\Lambda(X) \otimes \mathbb{Q}$ (and therefore its dual $\Lambda^*_{\mathbb{Q}}(X)$) with an inner product by restricting a W-invariant inner product on $X(T) \otimes \mathbb{Q}$ to $\Lambda(X) \otimes \mathbb{Q}$. As observed in [10], W_X does not depend on the choice of the W-invariant inner product on $X(T) \otimes \mathbb{Q}$.

Theorem 1. *(a) The valuation cone $V(X)$ is a simplicial cone: there exist linearly independent $\sigma_1, \sigma_2, \ldots, \sigma_s \in \Lambda(X)$ such that $V(X) = \{v \in \Lambda^*_{\mathbb{Q}}(X) \colon \langle v, \sigma_i \rangle \leq 0$ for all $i \in \{1, 2, \ldots, s\}\}$.*

*(b) The reflections over the codimension-one faces of $V(X)$ generate a finite subgroup W_X of $\operatorname{GL}(\Lambda^*_{\mathbb{Q}}(X))$. We call W_X the **little Weyl group** of X. In particular, $V(X)$ is a fundamental domain for the action of W_X on $\Lambda^*_{\mathbb{Q}}(X)$.*

(c) The lattice $\Lambda(X) \subset \Lambda(X) \otimes \mathbb{Q}$ is stable under the action of W_X on $\Lambda(X) \otimes \mathbb{Q}$. More precisely, W_X is a subgroup of the image of the map $N(\Lambda(X)) \to \operatorname{GL}(\Lambda^*_{\mathbb{Q}}(X))$ induced by the action of $N(\Lambda(X))$ on $\Lambda^*_{\mathbb{Q}}(X)$.

The theorem says that W_X is a crystallographic reflection group. Let $\Sigma(X)$ be the set of primitive elements $\sigma \in \Lambda(X)$ such that $\ker(\sigma) \subset \Lambda^*_{\mathbb{Q}}(X)$ is a wall of $V(X)$ and $\langle \sigma, V(X) \rangle \leq 0$. The elements of $\Sigma(X)$ are called the **spherical roots** of X. By construction, they are the simple roots of a root system in $\Lambda \otimes \mathbb{Q}$ with Weyl group W_X for which $V(X) \subset \Lambda^*_{\mathbb{Q}}(X)$ is the negative Weyl chamber. This definition is due to Luna [11, §1.2]. The set $\Sigma(X)$ of spherical roots of X is one of the three components of the 'spherical system' of X, a fundamental combinatorial invariant of X [11, §1.2 and §7.2]. For a given group G, the set $\{\sigma \in X(B) \colon \sigma$ is a spherical root of some spherical G-variety$\}$ is finite. If X is wonderful, then the set $\Sigma(X)$ has an elementary geometric description; see, e.g., [13, Definition 3.4.1].

Four other sets of simple roots for W_X. Other choices have been made with regards to the lengths of the simple roots associated to X. Let $\mathcal{L}$ be any $\mathbb{Z}$-submodule of $\Lambda(X) \otimes \mathbb{Q}$ generated by linearly independent vectors which satisfies the following two properties

(L1) $\mathcal{L}$ is W_X-stable; and
(L2) $\mathcal{L}^{\perp} := \{v \in \Lambda^*_{\mathbb{Q}}(X) \colon \langle v, \mathcal{L}\rangle = 0\}$ is contained in the linear part of $V(X)$.

Then the set $\Sigma(\mathcal{L})$ of primitive elements of $\mathcal{L}$ such that $\ker(\sigma) \subset \Lambda^*_{\mathbb{Q}}(X)$ is a wall of $V(X)$ and $\langle \sigma, V(X)\rangle \leq 0$ is also set of simple roots of a root system with Weyl group W_X.

Besides the standard choice $\mathcal{L} = \Lambda(X)$ mentioned above, four other natural choices are given below. We indicate afterwards why each $\mathcal{L}$ satisfies (L1) and (L2) and briefly discuss the role of each $\Sigma(\mathcal{L})$.

1. $\mathcal{L} = \Lambda(G/N_G(H))$; then $\Sigma(\mathcal{L})$ is denoted $\Sigma^N(X)$;
2. $\mathcal{L} = \Lambda(G/\overline{H})$, where $\overline{H}$ is the spherical closure (see below) of H; then $\Sigma(\mathcal{L})$ is denoted $\Sigma^{sc}(X)$;
3. $\mathcal{L} = \Lambda(X) \cap \Lambda_R = \Lambda(G/(ZH))$, where Λ_R is the root lattice of (G,T) and Z is the center of G; then $\Sigma(\mathcal{L})$ is denoted $\Sigma^K(X)$;
4. $\mathcal{L} = (\Lambda(X) \otimes \mathbb{Q}) \cap \Lambda_R$; then $\Sigma(\mathcal{L})$ is denoted $\Sigma^{SV}(X)$.

Recall that $N_G(H)$ acts on G/H by $n \cdot (gH) = gHn^{-1} = gn^{-1}H$. In fact, the induced map from $N_G(H)$ to the group of G-equivariant automorphisms of G/H is surjective and has kernel H, whence $\mathrm{Aut}^G(G/H) \cong N_G(H)/H$. It follows that $N_G(H)$ acts on the set $\mathcal{D}(G/H)$ of B-stable prime divisors (or colors, see [14]) of G/H. The kernel of this action, which contains H and Z, is called the **spherical closure** $\overline{H}$ of H. Luna introduced this notion and used it to reduce the classification of spherical varieties to that of wonderful varieties [11]. Knop proved that $G/\overline{H}$ has a wonderful compactification in [7, Corollary 7.6].

We now indicate why the four choices for $\mathcal{L}$ above satisfy (L1) and (L2). If K is a subgroup of G containing H then the surjection $G/H \to G/K$ implies that we have an inclusion $\Lambda(G/K) \subset \Lambda(G/H)$ and a surjective linear map $\pi \colon \Lambda^*_{\mathbb{Q}}(G/H) \to \Lambda^*_{\mathbb{Q}}(G/K)$. Moreover $\pi(V(G/H)) = V(G/K)$, see [5, §4]. One can show (using [5, Theorem 4.4] for example) that $\Lambda(G/N_G(H))^{\perp} \subset \Lambda^*_{\mathbb{Q}}(X)$ is the linear part of $V(X)$, which is also the invariant subspace of $\Lambda^*_{\mathbb{Q}}(X)$ for the action of W_X. It is now straightforward to show that if K is a subgroup of $N_G(H)$ containing H, then $\Lambda(G/K)$ satisifes (L1) and (L2). This takes care of the first three choices for $\mathcal{L}$. For the fourth choice, $\mathcal{L} = (\Lambda(X) \otimes \mathbb{Q}) \cap \Lambda_R$, condition (L1) follows from the second assertion in part (c) of Theorem 1. Condition (L2) follows from the fact that $\Lambda(X) \cap \Lambda_R$ satisfies it.

We briefly discuss the role of the four alternative sets of simple roots, in the same order as above.

1. $\Sigma^N(X)$: The subgroup $\Lambda(G/N_G(H)) \subset \Lambda(X)$ is the 'root lattice' of X, defined in [7, §6], and $\Sigma^N(X)$ is a basis of $\Lambda(G/N_G(H))$ and of the root system Δ_X Knop associates to X. If X is homogeneous or quasi-affine (see [7, Remark 6.6]), then the natural map $\mathrm{Aut}^G(X) \to \mathrm{Hom}(\Lambda(X), \mathbb{C}^\times)$ of [7, Theorem 5.4] induces an isomorphism $\mathrm{Aut}^G(X) \to \mathrm{Hom}(\Lambda(X)/\Lambda(G/N_G(H)), \mathbb{C}^\times)$. If X is quasi-affine, then there is a very simple construction of $\Sigma^N(X)$, see [7, Theorem 1.3]. This set also plays an important role in the geometry of Alexeev and Brion's moduli scheme of affine spherical varieties with a given weight monoid, see [1, Prop 2.13 and Cor 2.14].
2. $\Sigma^{sc}(X)$: We already mentioned the importance of the notion of spherical closure in Luna's classification program of spherical varieties. To be a bit more specific, his theory of augmentations allows one to combinatorially classify all spherical subgroups H of G with a given spherical closure [11, §6.4].
3. $\Sigma^K(X)$: This choice of normalization of the simple roots of W_X is the one in [8, §1]. In this paper, Knop defines the set of spherical roots of a spherical variety over a field of arbitrary characteristic and $\Sigma^K(X)$ is that set when the characteristic is zero.
4. $\Sigma^{SV}(X)$: This is the set of 'normalized simple spherical roots' of [15, §3.1], where the authors also conjecture that it is the set of simple roots of the 'dual group' of X defined by Gaitsgory and Nadler in [4].

From $\mathbf{\Sigma(X)}$ to $\mathbf{\Sigma^N(X)}$, $\mathbf{\Sigma^{sc}(X)}$ and $\mathbf{\Sigma^K(X)}$. The precise relationship between $\Sigma(X)$ and $\Sigma^N(X)$ was described by Losev in [9, Theorem 2]. Given $\sigma \in \Sigma(X)$, either $\sigma \in \Sigma^N(X)$ or $2\sigma \in \Sigma^N(X)$, and Losev's theorem says that $\sigma \in \Sigma(X)$ is doubled in $\Sigma^N(X)$ if and only if $\sigma \notin \Lambda_R$ or σ satisfies one of the conditions (1), (2) or (3) of [9, Definition 4.1.1]. The sets $\Sigma^{sc}(X)$ and $\Sigma^K(X)$ are obtained in a similar fashion from $\Sigma(X)$: for the latter one only doubles those $\sigma \in \Sigma(X)$ that do not belong to the root lattce Λ_R; for $\Sigma^{sc}(X)$ one doubles those $\sigma \in \Sigma(X)$ that do not belong to Λ_R or that satifsy condition (2) or (3) of [9, Definition 4.1.1].

Examples. The following examples were taken from [16]. For $X = \mathrm{SL}(2)/T$ one has $\Sigma(X) = \Sigma^{sc}(X) = \Sigma^K(X) = \Sigma^{SV}(X) = \{\alpha\}$ and $\Sigma^N(X) = \{2\alpha\}$, where α is the simple root of SL(2). For $X = (\mathrm{SL}(2) \times \mathrm{SL}(2))/\mathrm{SL}(2)$, we have $\Sigma(X) = \{\frac{\alpha+\alpha'}{2}\}$, whereas $\Sigma^N(X) = \Sigma^{sc}(X) = \Sigma^K(X) = \Sigma^{SV}(X) = \{\alpha + \alpha'\}$. When $X = \mathrm{SL}(3)/\mathrm{SO}(3)$ we have that $\Sigma^{SV}(X)$ is the set of simple roots of SL(3), whereas $\Sigma(X) = \Sigma^N(X) = \Sigma^{sc}(X) = \Sigma^K(X)$ consists of the doubles of the simple roots. Finally, when $X = \mathsf{G}_2/\mathrm{SL}(3)$, then $\Sigma(X) = \Sigma^K(X) = \Sigma^{SV}(X) = \{\alpha_1 + 2\alpha_2\}$, while $\Sigma^{sc}(X) = \Sigma^N(X) = \{2\alpha_1 + 4\alpha_2\}$, where α_1 and α_2 are the simple roots of G_2.

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