Matrix Coefficients and Linearization Formulas for SL(2)

Robert W. Donley, Jr.
(Queensborough Community College)

November 17, 2017

Goals of Talk

1. Review of last talk
2. Special Functions
3. Matrix Coefficients
4. Physics Background
5. Matrix calculator for $c_{m,n,k}(i,j)$
   (Vanishing of $c_{m,n,k}(i,j)$ at certain parameters)

References

1. Andrews, Askey, and Roy, Special Functions (big red book)
2. Vilenkin, Special Functions and the Theory of Group Representations (big purple book)
3. Beiser, Concepts of Modern Physics, 4th edition
4. Donley and Kim, "A rational theory of Clebsch-Gordan coefficients," preprint. Available on arXiv

Review of Last Talk

$X = SL(2, \mathbb{C})/T$

$n \geq 0:$ $V(2n)$ highest weight space for highest weight $2n$,
$\dim(V(2n)) = 2n + 1$

$$\mathbb{C}[SL(2,\mathbb{C})/T] \cong \sum_{n \in \mathbb{N}} V(2n)$$

$$^{T}\mathbb{C}[SL(2,\mathbb{C})/T] = \sum_{n \in \mathbb{N}} \mathbb{C}\, f_{2n}$$

f_{2n} is called a **zonal spherical function** of type $2n$.
That is, $T \cdot f_{2n} = f_{2n}$.

Linearization Formula

1) Weight 0 : $t \cdot (f_{2m}\, f_{2n}) = (t \cdot f_{2m})(t \cdot f_{2n}) = f_{2m}\, f_{2n}$

2) $f_{2m}\, f_{2n} \in V(2m) \otimes V(2n) \cong \sum_{k=0}^{\min(m,n)} V(2m+2n-2k)$

(Clebsch-Gordan decomposition)

That is, $f_{2m}\, f_{2n}$ is also spherical and a finite sum of zonal spherical functions.

$$f_{2m} \cdot f_{2n} = \sum_{p=|m-n|}^{m+n} c(2m, 2n, 2p)\ f_{2p}$$

Gap 4

1. For what m, n, p is $c(2m, 2n, 2p)$ nonzero?
2. Explicit calculation of $c(2m, 2n, 2p)$.

Main result (Gap 4)

$$f_{2m} \cdot f_{2n} = \sum_{k=0}^{\min(m,n)} c(2m, 2n, 2m+2n-4k)\ f_{2m+2n-4k}$$

Compare with Clebsch-Gordan Decomposition:

$$V(2m) \otimes V(2n) \cong V(2m+2n) \oplus V(2m+2n-2) \oplus \cdots \oplus V(|2m-2n|)$$

$$f_{2m} f_{2n} \in V(2m+2n) \oplus V(2m+2n-4) \oplus \cdots \oplus V(|2m-2n|)$$

Special Functions Background

Orthogonal Polynomials

Basis $\{p_n(x)\}_{n=0}^{\infty}$ of $\mathbb{C}[x]$ such that

1. $p_0(x) = 1$
2. $deg(p_n(x)) = n$, and
3. $\int_a^b p_m(x)p_n(x)d\mu(x) = h_n\delta_{m,n}$

for some positive measure $d\mu(x)$ on $[a, b]$.

Recursion:

$$xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x)$$

Jacobi Polynomials $P_n^{(\alpha,\beta)}(x)$

"Rodrigues" Formula:

$$(1-x)^\alpha(1+x)^\beta \; P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}[(1-x)^{n+\alpha}(1+x)^{n+\beta}]$$

Positive measure:

$$d\mu = (1-x)^\alpha(1+x)^\beta dx \qquad \text{on} \qquad [-1,1]$$

Formula for h_n

$$\int_{-1}^{1} P_m^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(x)\; d\mu = \frac{2^{\alpha+\beta+1}\;\Gamma(n+\alpha+1)\;\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\;\Gamma(n+\alpha+\beta+1)\;n!}\;\delta_{m,n}$$

Legendre Polynomials $P_n(x) = P_n^{(0,0)}(x)$

Rodrigues Formula (1816):

$$P_n(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n}[(1-x^2)^n]$$

Positive measure:

$$d\mu = dx \qquad \text{on} \qquad [-1,1]$$

Formula for h_n

$$\int_{-1}^{1} P_m(x)\ P_n(x)\ d\mu = \frac{2}{(2n+1)}\ \delta_{m,n}$$

Linearization Formula for $P_n(x)$

Ferrers (1877), Adams (1877), independently:

$$P_m(x) \cdot P_n(x) = \sum_{k=0}^{\min(m,n)} a_{m,n,k} P_{m+n-2k}(x)$$

with

$$a_{m,n,k} = \frac{2m+2n+1-4k}{2m+2n+1-2k} \cdot \frac{(1/2)_k (1/2)_{m-k} (1/2)_{n-k} (m+n-k)!}{k!\ (m-k)!\ (n-k)!\ (1/2)_{m+n-k}}$$

Shifted factorial: $(x)_k = x(x+1)(x+2)\ldots(x+k-1)$

Gaps: $P_n(x)$ has same parity as n (even/odd as functions)

Note: $a_{m,n,k} > 0$

Representations to Functions

Compact picture: $SU(2)$ has same representation theory as $SL(2,\mathbb{C})$.

Recall: $(\pi_{2n}, V(2n))$ can be realized as $(L, \mathbb{C}[x,y]_{2n})$

with $SU(2)$-invariant inner product $\langle\cdot,\cdot\rangle_{2n}$.

Since $\pi_{2n}(-I) = id$ for $-I \in SU(2) \subset SL(2,\mathbb{C})$,

$(\pi_{2n}, V(2n))$ admits an action of $SO(3)$ since $SO(3) \cong SU(2)/\pm I$

Implement by Adjoint map:
$k \in SU(2)$ acts on traceless skew-Hermitian matrices of size 2 by

$$Ad: \; SU(2) \to SO(sSHerm) \cong SO(3),$$

$$Ad(k): \; sSHerm \to sSHerm, \qquad Ad(k)X = kXk^{-1}$$

Geometry: $S^3 \to \mathbb{R}P^3$

Fourier Analysis

Features of $L^2(SO(3))$:

1. joint action by $SO(3) \times SO(3)$: $(\pi(k_1, k_2)f)(g) = f(k_1^{-1} g k_2)$
2. bi-invariant Haar measure dk: $\int_{SO(3)} \, dk = 1$,

$$\int_{SO(3)} f(k_1 k k_2) \, dk = \int_{SO(3)} f(k) \, dk,$$

3. (Peter-Weyl) $L^2(SO(3)) \cong \sum V(2n) \otimes V(2n)^*$

Compare with $K = S^1$ acting on

$$L^2(S^1) \cong \sum_{n \in \mathbb{Z}} \mathbb{C} e^{in\theta}$$

Matrix coefficients

To pass from representations to functions in $L^2(SO(3))$,

Definition:

Choose u, v in $V(2n)$.
Define the **matrix coefficient**

$$\phi_{u,v} : SO(3) \to \mathbb{C}$$

by

$$\phi_{u,v}(k) = \langle \pi_{2n}(k)u, v \rangle.$$

This includes

$$V(2n) \otimes V(2n)^* \to L^2(SO(3)) \cong \sum V(2m) \otimes V(2m)^*$$

as an irreducible $SO(3) \times SO(3)$ representation.

Schur Orthogonality

Key formula (Schur orthogonality):

If u, v, u', v' in $V(2n)$, then

$$\int_{SO(3)} \phi_{u,v}(k)\, \overline{\phi_{u',v'}(k)}\, dk = \frac{1}{2n+1} \langle u, u' \rangle \overline{\langle v, v' \rangle}.$$

If u, v in $V(2m)$ and u', v' in $V(2n)$ with $m \neq n$ then

$$\int_{SO(3)} \phi_{u,v}(k)\, \overline{\phi_{u',v'}(k)}\, dk = 0.$$

ONB for each $V(2n)$ $\quad \rightarrow \quad$ ONB of matrix coefficients for $L^2(SO(3))$
That is,

$$f(k) = \sum c_{2n,i,j}\, \sqrt{2n+1}\, \phi_{u_i,u_j}(k)$$

Summands in $L^2(SO(3)/T)$

For the summand of "$V(2n)$" in

$$L^2(SO(3)/T) \cong \sum \text{"}V(2m)\text{"},$$

fix any abstract realization of $(\pi_{2n}, V(2n))$,

fix a nonzero spherical vector ϕ_0 in $V(2n)$ and any v in $V(2n)$.

Then "$V(2n)$" is spanned by the matrix coefficients

$$\phi_{v,\phi_0}(k) = \langle v, \pi_{2n}(k)\phi_0 \rangle.$$

with group action

$$L(g)\phi(k) = \phi(g^{-1}k)$$

Tensors

Consider $V(2m) \otimes V(2n)$:
Invariant inner product:

$$\langle u \otimes v, u' \otimes v' \rangle := \langle u, u' \rangle \langle v, v' \rangle$$

If we multiply matrix coefficients,

$$\phi^{2m}_{u,u'}(k)\ \phi^{2n}_{v,v'}(k) = \phi^{\otimes}_{u \otimes v, u' \otimes v'}(k)$$

Linearization:

$$\phi^{2m}_{u,u'}(k)\ \phi^{2n}_{v,v'}(k) = \sum_{p} c(2m, 2n, 2p, i, j)\ \phi^{2p}_{u_i,u_j}(k)$$

Zonal spherical functions

Now choose unit spherical vectors v_0^{2m} in $V(2m)$ (resp. $2n$)

If we multiply,

$$\phi^{2m}_{v_0^{2m},v_0^{2m}}(k)\ \phi^{2n}_{v_0^{2n},\phi_0^{2n}}(k) = \phi^{\otimes}_{v_0^{2m}\otimes v_0^{2n},v_0^{2m}\otimes v_0^{2n}}(k)$$

If

$$v_0^{2m} \otimes v_0^{2n} = \sum c(2m,2n,2p)\ v_0^{2p}$$

then

$$\phi^{2m}_{v_0^{2m},v_0^{2m}}(k)\ \phi^{2n}_{v_0^{2n},\phi_0^{2n}}(k) = \sum\ |c(2m,2n,2p)|^2\ \phi^{2p}_{v_0^{2p},v_0^{2p}}(k)$$

Zonal spherical functions as Matrix coefficients

Restrict to one-parameter subgroup: $L^2_{cont}(SO(3)) \to L^2_{cont}(S^1)$

$$\begin{pmatrix} \cos(\theta/2) & i\sin(\theta/2) \\ i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$\phi^{2m}_{v_0^{2m}, v_0^{2m}}(\theta) = P_m^{(0,0)}(\cos(\theta))$$

Physics Background

Hydrogen Atom: Bound Electron, Discrete Binding Energy E_n

$\psi(x, y, z)$ probability wave function of electron

Time-Independent Schrödinger Wave Equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} + \frac{2m}{\hbar^2}(E_n - V)\psi = 0$$

where

$$V = -\frac{e^2}{4\pi\epsilon_0 r} \qquad \text{and} \qquad E_n = -\frac{me^4}{32\pi^2\epsilon_0^2\hbar^2 n^2} = \frac{E_1}{n^2}$$

Solution Space

Solve using
1) rotation invariance (spherical coordinates) and
2) separation of variables.

$$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$$

where

$$\Phi(\phi) = Ae^{im_l\phi},$$

$$\frac{1}{\sin(\theta)}\frac{d}{d\theta}\Big(\sin(\theta)\frac{d\Theta}{d\theta}\Big) \;+\; \Big[l(l+1) - \frac{m_l^2}{\sin^2(\theta)}\Big]\Theta = 0$$

$$\frac{1}{r^2}\frac{d}{dr}\Big(r^2\frac{dR}{dr}\Big) + \Big[\frac{2m}{\hbar^2}\Big(\frac{e^2}{4\pi\epsilon_0 r} + E_n\Big) - \frac{l(l+1)}{r^2}\Big]R = 0$$

Quantum Numbers

$$\psi_{n,l,m_l} = R_{n,l}\ \Theta_{l,m_l}\ \Phi_{m_l}$$

Principal Quantum Number $= n = 1, 2, 3, \ldots$

Orbital Quantum Number $= l = 0, 1, 2, 3, \ldots n - 1$

Magnetic Quantum Number $= m_l = 0, \pm 1, \pm 2, \pm 3, \ldots, \pm l$

Linear Algebra/Representation Theory of $SO(3)$

E_n is binding energy in original differential equation.
Total energy is determined by n and l.

Orbital quantum number l corresponds to the magnitude L of the electron's angular momentum $\mathbf{L}$:

$$|\mathbf{L}|^2 = l(l+1)\hbar^2.$$

(Representation theory: highest weight)

Magnetic quantum numbers correspond to eigenvalues (weights) for projection in z-direction,

$$\mathbf{L}_z = m_l \hbar.$$

Thus, l determines an $l+1$ dimensional space.
m_l determines a one-dimensional eigenspace under $\mathbf{L}_z$.

Coupling of Angular Momentum States

If two electrons are combined into one system, multiply wave functions.

$$\Phi_{m_{l_1}}(\phi) \cdot \Phi_{m_{l_2}}(\phi) = A_1 e^{i m_{l_1} \phi} A_2 e^{i m_{l_2} \phi} = A_3 \Phi_{m_{l_1}+m_{l_2}}(\phi)$$

$$\Theta_{l_1, m_{l_1}}(\theta) \cdot \Theta_{l_2, m_{l_2}}(\theta) = \sum_i C(l_1, l_2, m_{l_1}, m_{l_2}, l_i)\ \Theta_{l_i, m_{l_1}+m_{l_2}}(\theta)$$

$C(l_1, l_2, m_{l_1}, m_{l_2}, l_i)$ is called a Clebsch-Gordan coefficient.

Restriction on m_{l_i}: $\quad -l_1 - l_2 \le m_{l_1} + m_{l_2} \le l_1 + l_2$

Restriction on l_i: $\quad |l_1 - l_2| \le l_i \le l_1 + l_2$

(Think $\sin(l_1\theta) \cdot \sin(l_2\theta) = \frac{1}{2}(\cos((l_1 - l_2)\theta) - \cos((l_1 + l_2)\theta)))$

Wigner's Formula (1931)

(Note: different indexing from quantum numbers)

$$C'_{m,n,k}(i,j) = \sqrt{\frac{(m+n-2k+1)(m+n-i-j-k)!(i+j-k)!i!j!k!}{(m-i)!(n-j)!(m-k)!(n-k)!(m+n-k+1)!}} \cdot S$$

where

$$S = \sum_{s=0}^{k} (-1)^{i-s} \frac{(m-i+s)!(n-k+i-s)!}{s!(i+j-k-s)!(i-s)!(k-i+s)!}$$

Traditionally studied through representation theory, Jacobi polynomials and hypergeometric series of type ${}_3F_2$

Wigner's Formula (1931)

With binomial coefficients and shift $s \to i - s$,

$$C'_{m,n,k}(i,j) = \sqrt{\frac{(m+n-2k+1)\binom{m+n-k}{m-i,\ n-j,\ *}}{(m+n-k+1)\ \binom{m+n-k}{m-k,\ n-k,\ k}\binom{m+n-k}{i,\ j,\ *}}} \cdot S$$

where

$$S = \sum_{s=0}^{k} (-1)^s \binom{i+j-k}{i-s}\binom{m-s}{k-s}\binom{n-k+s}{s}$$

Racah's Formula (1942)

$$C'_{m,n,k}(i,j) = \sqrt{\frac{(m+n-2k+1)\binom{m+n-k}{m-k,\ n-k,\ k}}{(m+n-k+1)\ \binom{m+n-k}{m-i,\ n-j,\ *}\binom{m+n-k}{i,\ j,\ *}}} \cdot S'$$

where

$$S' = \sum_{s=0}^{k} (-1)^s \binom{k}{s}\binom{m-k}{i-s}\binom{n-k}{j-k+s}.$$

Pascal's Recurrence

Immediate Goal: Abandon Formula for Matrix Calculator

$$c_{m,n,k}(i,j) = \sum_{s=0}^{k} (-1)^s \binom{i+j-k}{i-s} \binom{m-s}{k-s} \binom{n-k+s}{s}$$

Apply Pascal's Identity

$$\binom{i+j-k}{i-s} = \binom{(i-1)+j-k}{(i-1)-s} + \binom{i+(j-1)-k}{i-s}$$

to the first binomial coefficent to get

$$c_{m,n,k}(i,j) = c_{m,n,k}(i-1,j) + c_{m,n,k}(i,j-1)$$

The Algorithm for the Matrix $M(m, n, k)$

Given non-negative $m,\ n,\ k$ with $0 \leq k \leq min(m, n)$

1. Initialize matrix of size $(m+1)$ by $(m+n-2k+1)$,
2. Initialize left-most column and top row with binomial coefficient pattern, (hold for last)
3. Pascal's Recurrence to the Right (L-pattern),
4. Zero out upper-right column for symmetry.

m=3, n=4, k=0

$(m+1)\ x\ (m+n-2k+1)\ =\ 4\ x\ 8$

$k=0$: Pascal's Triangle (truncate at right corner for symmetry)

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & 0 & 0 \\ 0 & 0 & 1 & 3 & 6 & 10 & 15 & 0 \\ 0 & 0 & 0 & 1 & 4 & 10 & 20 & 35 \end{bmatrix}$$

m=3, n=4, k=1:

$(m+1)\ x\ (m+n-2k+1) \ = \ 4\ x\ 6$

$k=1$: Set left-most column - $k+1=2$ nonzero entries,
Pascal's Recurrence to Right

$$\begin{bmatrix} 3 & 3 & 3 & 3 & 0 & 0 \\ -4 & -1 & 2 & 5 & 8 & 0 \\ 0 & -4 & -5 & -3 & 2 & 10 \\ 0 & 0 & -4 & -9 & -12 & -10 \end{bmatrix}$$

m=3, n=4, k=2

$(m+1)\ x\ (m+n-2k+1)\ =\ 4\ x\ 4$

$k = 2$: Set left-most column - $k+1 = 3$ nonzero entries, Pascal's Recurrence to Right

$$\begin{bmatrix} 3 & 3 & 3 & 0 \\ -6 & -3 & 0 & 3 \\ 6 & 0 & -3 & -3 \\ 0 & 6 & 6 & 3 \end{bmatrix}$$

m=3, n=4, k=3

$(m+1)$ x $(m+n-2k+1)$ = 4 x 2

$k = 3$: Set left-most column - $k+1 = 4$ nonzero entries,
Pascal's Recurrence to Right

$$\begin{bmatrix} 1 & 1 \\ -2 & -1 \\ 3 & 1 \\ -4 & -1 \end{bmatrix}$$

First $k+1$ Entries Of The Left-Most Column

First column has $i + j - k = 0$ so first binomial coefficient $= 1$

$$(-1)^s \binom{m-s}{k-s} \binom{n-k+s}{s}, \qquad 0 \le s \le k$$

$$k = 1 : \quad \left(\binom{3}{1}\binom{3}{0}, -\binom{2}{0}\binom{4}{1} \right) \to (3, -4, 0, 0)$$

$$k = 2 : \quad \left(\binom{3}{2}\binom{2}{0}, -\binom{2}{1}\binom{3}{1}, \binom{1}{0}\binom{4}{2} \right) \to (3, -6, 6, 0)$$

$$k = 3 : \quad \left(\binom{3}{3}\binom{1}{0}, -\binom{2}{2}\binom{2}{1}, \binom{1}{1}\binom{3}{2}, -\binom{0}{0}\binom{4}{3} \right)$$

$$\to (1, -2, 3, -4)$$

First Main Result

Fix m, n, k as before. The $(i+1, i+j-k+1)$ entry of the corresponding matrix $M(m,n,k)$ in the algorithm equals

$$c_{m,n,k}(i,j) = \sum_{s=0}^{k} (-1)^s \binom{i+j-k}{i-s} \binom{m-s}{k-s} \binom{n-k+s}{s}$$

Pascal's Recurrence:

$$c_{m,n,k}(i,j) = c_{m,n,k}(i-1,j) + c_{m,n,k}(i,j-1)$$

Weyl Group Symmetry

$$c_{m,n,k}(m-i, n-j) = (-1)^k \frac{(m-i)!(n-j)!(i+j-k)!}{i!\, j!\, (m+n-i-j-k)!} c_{m,n,k}(i,j)$$

Net effect:

1. rotate $M(m,n,k)$ by 180 degrees,
2. change signs by parity of k, and
3. rescale entries by positive scalar (rational function of five parameters).

Important effect: zero locus is preserved by rotation by 180 degrees

Reverse Recurrence

$$a_1 \; c_{m,n,k}(i,j) \;=\; a_2 \; c_{m,n,k}(i+1,j) \;+\; a_3 \; c_{m,n,k}(i,j+1)$$

where

1. $a_1 = (i+j-k+1)(m+n-i-j-k)$,
2. $a_2 = (i+1)(m-i)$,
3. $a_3 = (j+1)(n-j)$.

Key Points:

1. upside-down L-pattern, and
2. coefficients are positive when $i < m$, $j < n$.

Central Value/Central Square

Suppose m and n are even. Then $M(m, n, k)$ has odd side lengths $m + 1$ by $m + n - 2k + 1$.

There is a middle entry $c_{m,n,k}(\frac{m}{2}, \frac{n}{2})$,
which we call the **central value**.

We call the adjacent values the **central square**, a 3x3 submatrix.

Censorship Rule

When $k < \min(m, n)$, all proper zeros in $M(m, n, k)$ are of the form

$$\begin{pmatrix} \blacksquare & \blacksquare & * \\ \blacksquare & 0 & \blacksquare \\ * & \blacksquare & \blacksquare \end{pmatrix}$$

where $\blacksquare$ must be non-zero.

Proof: If adjacent zeros, use both recurrences:

$$\begin{pmatrix} * & * & * \\ * & * & * \\ * & 0 & 0 \end{pmatrix} \to \begin{pmatrix} * & * & * \\ * & 0 & * \\ * & 0 & 0 \end{pmatrix} \to \begin{pmatrix} * & * & * \\ 0 & 0 & * \\ * & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ * & 0 & 0 \end{pmatrix},$$

eventually placing a zero on top row in a nonzero position.

Vanishing Central Value Rule (Gap 4)

Proposition: Suppose m and n are even. The central value $c_{m,n,k}(\frac{m}{2}, \frac{n}{2}) = 0$ if and only if k is odd.

Proof: If k is odd, the Weyl group symmetry gives

$$c_{m,n,k}(\frac{m}{2}, \frac{n}{2}) = (-1)^k c_{m,n,k}(\frac{m}{2}, \frac{n}{2}) = -c_{m,n,k}(\frac{m}{2}, \frac{n}{2}).$$

so

$$c_{m,n,k}(\frac{m}{2}, \frac{n}{2}) = 0.$$

Other direction

We have a central square with nonzero X and Y

$$\begin{pmatrix} * & * & * \\ * & 0 & * \\ * & * & * \end{pmatrix} \to \begin{pmatrix} -Y & * & * \\ Y & 0 & * \\ * & X & X \end{pmatrix}$$

Reverse recurrence on lower left hook implies $a_1 Y = a_2 X$
so X and Y have same parity.

But Weyl group symmetry changes X to $-Y$ so k must be odd.
$((-1)^k = -1)$

Back to representations

Choose highest weight vectors ϕ_m, ϕ_n for $V(m)$ and $V(n)$

Basis for $V(m)$: $\{f^i \phi_m\}$

Basis for $V(m) \otimes V(n)$: $\{f^i \phi_m \otimes f^j \phi_n\}$

Basis for $V(m+n-2k) \subset V(m) \otimes V(n)$: $\{f^{l-k} \phi_{m,n,k}\}$

$$\phi_{m,n,k} = \sum_i c_{m,n,k}(i, k-i) f^i \phi_m \otimes f^{k-i} \phi_n$$

$$f^{l-k} \phi_{m,n,k} = \sum_{i+j=l} c_{m,n,k}(i,j) f^i \phi_m \otimes f^j \phi_n$$

Back to representations

Leibniz Rule:

$$f \cdot (f^i \phi_m \otimes f^j \phi_n) = f^{i+1} \phi_m \otimes f^j \phi_m + f^i \phi_m \otimes f^{j+1} \phi_n$$

implies

$$c_{m,n,k}(i,j) = c_{m,n,k}(i-1,j) + c_{m,n,k}(i,j-1)$$

Gap 4

Interpret: $m = 2m', n = 2n'$,

$i = m', j = n'$ implies weight 0 (spherical)

Nonvanishing for even $k = 2k'$implies

$$f^{m'}\phi_m \otimes f^{n'}\phi_n = \sum c_{m,n,2k'}(m', n')f^{m'+n'-2k'}\phi_{m,n,2k'}$$

with $\phi_{m,n,2k'} \in V(m+n-4k')$